Technical Memorandum

The Discrete-Time Fourier Transform and Discrete Fourier Transform of Windowed Stationary White Noise

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1 Introduction

We derive expressions for the mean, mean-square, and variance, of the discrete-time Fourier transform (DTFT) and K-point discrete Fourier transform (DFT) of a realization $x[n]$ of an N-point complex-valued stationary random process. We require $K \geq N$. For the case where $x[n]$ is also Gaussian and white, we also compute the probability density function (PDF) of the DTFT and DFT. $x[n]$ is not restricted to be zero mean; it may have a non-zero, complex mean μ. Thus we can decompose $x[n]$ into a constant and a zero-mean part:

$$x[n] = \mu + \tilde{x}[n] \tag{1}$$

We denote the variance of $x[n]$ as σ_x^2.In addition, we will allow for the use of a real-valued N-point window function $w[n]$ in computing the DTFT or DFT.[1] The windowed sequence is denoted $x_w[n] = w[n]x[n]$.

1.1 Summary of Main Results

The following two tables summarize the main results derived in this memo. In these tables, $X_w(\omega)$, $X(\omega)$, and $W(\omega)$ are the DTFTs of $x_w[n]$, $x[n]$, and $w[n]$, respectively. $X_w[k]$, $X[k]$, and $W[k]$ are the DFTs of $x_w[n]$, $x[n]$, and $w[n]$, respectively. μ is the mean of $x[n]$, and the symbol $\otimes_\omega$ denotes circular convolution in frequency on the interval $[-\pi,\pi]$, and the symbol $\otimes_k$ denotes discrete circular convolution in frequency on the interval $[0,K-1]$. $S_{\tilde{x}}(\omega)$ and $S_{\tilde{x}}[k]$ are the continuous- and discrete-frequency power spectra of $\tilde{x}[n]$. E_w is the energy in the window function $w[n]$. Finally, $\gamma_r(\omega)$ and $\gamma_r(\omega)$ are the (frequency-dependent) means of the real and imaginary parts of $X_w(\omega)$, which we denote $X_{rw}(\omega)$ and $X_{rw}(\omega)$; see Eqn. (32) for the value of $\gamma_r(\omega)$ and $\gamma_r(\omega)$.

[1] Most of the results obtained would be the same if we allowed the window to be complex, but that case is not often of interest.

Table 1. Probability Density Function of the DTFT or K-point ($K \geq N$) DFT of N-point Complex White Random Signal $x[n]$ with Real Window $w[n]$. γ_r and γ_I are frequency-dependent.

Case	PDF
Non-zero mean $x[n]$***, equal real and imaginary variance***	
$X_{wr}(\omega)$ or $X_{wr}[k]$	$\frac{1}{\sqrt{\pi E_w \sigma_x^2}} \exp\left[-\left(X_{wr}-\gamma_r\right)^2 \big/ E_w \sigma_x^2\right]$
$X_{wi}(\omega)$ or $X_{wi}[k]$	$\frac{1}{\sqrt{\pi E_w \sigma_x^2}} \exp\left[-\left(X_{wi}-\gamma_i\right)^2 \big/ E_w \sigma_x^2\right]$
$\|X_w(\omega)\|$ or $\|X_w[k]\|$	$\begin{cases} \frac{2\|X\|}{E_w \sigma_x^2} \exp\left[-\left(\|X\|^2+\|\gamma\|^2\right) \big/ E_w \sigma_x^2\right] I_0\left(\frac{2\|X\|\|\gamma\|}{E_w \sigma_x^2}\right), & \|X\| \geq 0 \\ 0, & \text{otherwise} \end{cases}$
$Y \equiv \|X_w(\omega)\|^2$ or $\|X_w[k]\|$	$\begin{cases} \frac{1}{E_w \sigma_x^2} \exp\left[-\left(Y+\|\gamma\|^2\right) \big/ E_w \sigma_x^2\right] I_0\left(\frac{2Y\|\gamma\|}{E_w \sigma_x^2}\right), & Y \geq 0 \\ 0, & \text{otherwise} \end{cases}$
$\arg\{X_w(\omega)\}$ or $\arg\{X_w[k]\}$	see text
Zero mean $x[n]$***, equal real and imaginary variance***	
$X_{wr}(\omega)$ or $X_{wr}[k]$	$\frac{1}{\sqrt{\pi E_w \sigma_x^2}} \exp\left[-X_{wr}^2 \big/ E_w \sigma_x^2\right]$
$X_{wi}(\omega)$ or $X_{wi}[k]$	$\frac{1}{\sqrt{\pi E_w \sigma_x^2}} \exp\left[-X_{wi}^2 \big/ E_w \sigma_x^2\right]$
$\|X_w(\omega)\|$ or $\|X_w[k]\|$	$\begin{cases} \frac{2\|X\|}{E_w \sigma_x^2} \exp\left[-\|X\|^2 \big/ E_w \sigma_x^2\right], & \|X\| \geq 0 \\ 0, & \text{otherwise} \end{cases}$
$Y \equiv \|X_w(\omega)\|^2$ or $\|X_w[k]\|$	$\begin{cases} \frac{1}{E_w \sigma_x^2} \exp\left[-Y \big/ E_w \sigma_x^2\right], & Y \geq 0 \\ 0, & \text{otherwise} \end{cases}$
$\arg\{X_w(\omega)\}$ or $\arg\{X_w[k]\}$	$\begin{cases} \frac{1}{2\pi}, & -\pi \leq \phi \leq \pi \\ 0, & \text{otherwise} \end{cases}$

Table 2. Mean and Variance for DTFT and K-point ($K \ge N$) DFT of N-point Complex Random Signal $x[n]$ with Real Window $w[n]$.

Case	Mean	Variance
Properties of DTFT		
General $x[n]$	$\mu W(\omega)$	$\frac{1}{2\pi}\left\|W(\omega)\right\|^2 \otimes_\omega S_{\tilde{x}}(\omega)$
White $x[n]$	$\mu W(\omega)$	$E_w \sigma_{\tilde{x}}^2$
White $x[n]$, no window	$\mu \frac{\sin[\omega N/2]}{\sin[\omega/2]} e^{-j\omega(N-1)/2}$	$N\sigma_{\tilde{x}}^2$
Properties of DFT		
General $x[n]$	$\mu W[k]$	$\frac{1}{K}\left\|W[k]\right\|^2 \otimes_k S_{\tilde{x}}[k]$
White $x[n]$	$\mu W[k]$	$E_w \sigma_{\tilde{x}}^2$
White $x[n]$, no window	$\mu \frac{\sin[\pi k N/K]}{\sin[\pi k/K]} e^{-j\pi k(N-1)/K}$	$N\sigma_{\tilde{x}}^2$
White $x[n]$, no window, $K = N$	$N\mu\delta[k]$	$N\sigma_{\tilde{x}}^2$

1.2 Fourier Transform Relationship:

Recall that the definition of the DTFT is [1]

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \tag{2}$$

Since we assume $x[n]$ has finite support on $[0,N-1]$, this becomes for our purposes

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} \tag{3}$$

The definition of the K-point DFT for a sequence $x[n]$ of length $N \le K$ is[2]

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/K}, \quad 0 \le k \le K-1 \tag{4}$$

Note that for the conditions given, namely $x[n]$ of finite length $N \le K$, we have

[2] The case where the DFT is sampled at $K < N$ frequencies requires the definition of an aliased version of $x[n]$ for use in (4); this technique is called *data turning* [2]. Many, though not all, of the formulas in this memo still apply in this case. However, it is unusual to have $K < N$ so, in order to have one less complication, we do not include that case here.

$$X[k] = X(\omega)\Big|_{\omega=\frac{2\pi k}{K}}, \quad 0 \le k \le K-1 \tag{5}$$

Thus, the DFT samples a DTFT in frequency.

2 Mean

2.1 Mean of $X_w(\omega)$

The mean of $X_w(\omega)$ is

$$\begin{aligned}\overline{X_w(\omega)} &= \overline{\sum_{n=0}^{N-1} w[n]x[n]e^{-j\omega n}} = \sum_{n=0}^{N-1} \overline{w[n]x[n]e^{-j\omega n}} = \sum_{n=0}^{N-1} w[n]\overline{x[n]}e^{-j\omega n} \\ &= \mu \sum_{n=0}^{N-1} w[n]e^{-j\omega n} \\ &= \mu W(\omega)\end{aligned} \tag{6}$$

where the overbar indicates expected value. Thus, a nonzero mean in the input data contributes a nonzero mean term at *every* frequency in the DTFT, not just at “DC” ($\omega = 0$). This is not surprising; the result is simply the DTFT of the finite-length constant signal corresponding to the input mean. The relative values of the mean at different frequencies are determined by the window spectrum. Note that if the input process mean $\mu = 0$, the mean of the DTFT is zero at all frequencies ω and in all cases of window shape.

The case where $w[n] = 1$, $0 \le n \le N-1$ (referred to here as the “no window” case) is worth special mention. In this case it is easy to show that [1]

$$W(\omega) = \frac{\sin[\omega N/2]}{\sin[\omega/2]} e^{-j\omega(N-1)/2} \tag{7}$$

Thus the mean of $X_w(\omega)$ is a scaled “asinc” or Dirichlet function.

2.2 Mean of $X_w[k]$

The mean of $X_w[k]$ is readily obtained by using (5) in (6):

$$\overline{X_w[k]} = \mu W\left(\frac{2\pi k}{K}\right) = \mu W[k] \tag{8}$$

In the “no window” case of $w[n] = 1$, $0 \le n \le N$-1, this becomes, using (7),

$$\overline{X[k]} = \mu \frac{\sin[\pi k N/K]}{\sin[\pi k/K]} e^{-j\pi k(N-1)/K} \quad (\text{no window}) \tag{9}$$

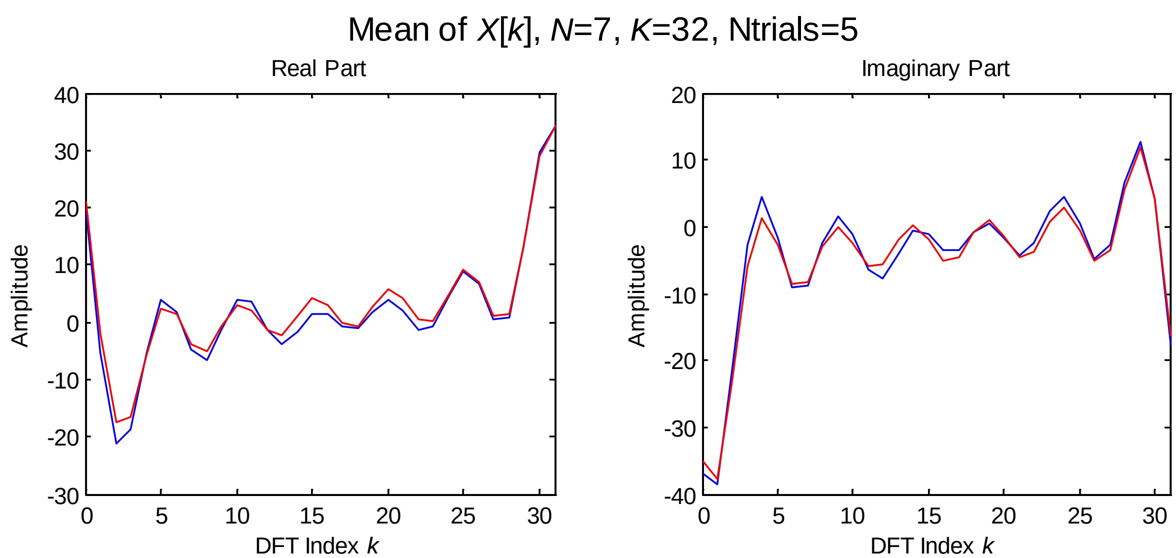
Note that (9) is indeterminate at $k = 0$; L'Hospital's rule or direct computation of the $k = 0$ case gives $\left|\overline{X[0]}\right| = N\mu$. If in addition $K = N$, Eqn. (9) simplifies further to

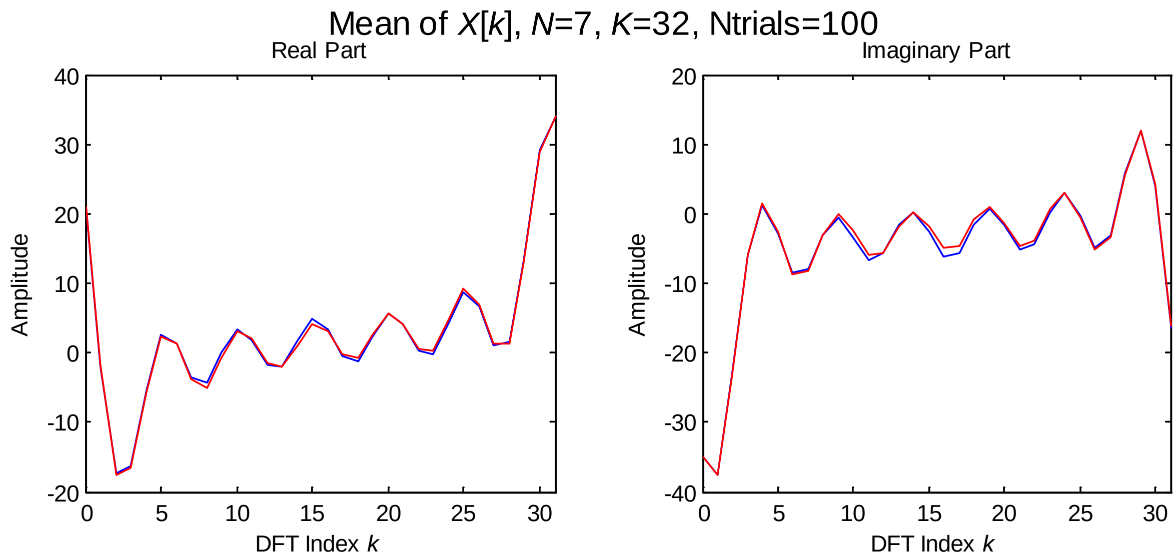
$$\overline{X[k]} = N\mu\delta[k] = \begin{cases} N\mu, & k = 0 \quad (\text{"DC term"}) \\ 0, & k = 1, \ldots, N-1 \end{cases} \qquad (K = N, \quad \text{no window}) \tag{10}$$

This occurs because when $K = N$, the DFT samples the underlying DTFT at the zero crossings of the asinc function, except for the sample at $k = 0$, corresponding to $\omega = 0$. Thus in this case, a nonzero mean input results in a nonzero mean of the DFT only at the DC term, $k = 0$. If $K > N$ or a non-trivial window is used, all of the DFT samples will, in general, have a non-zero mean.

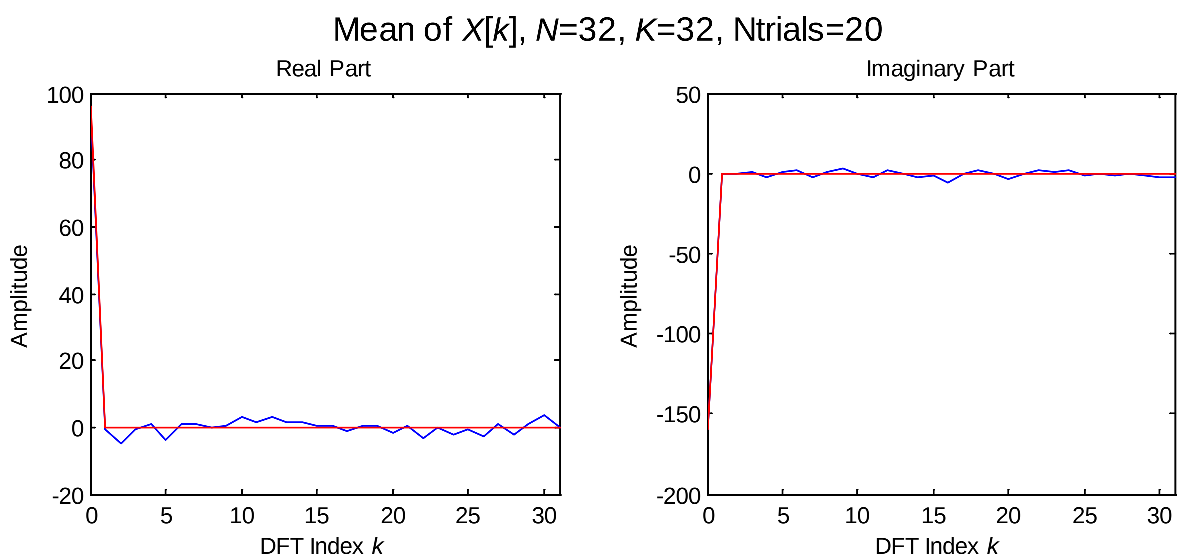
Again, note that if the input process mean $\mu = 0$, the mean of the DFT is zero at all frequency indices k and in all cases of window shape and length.

The DFT results are illustrated in the following figures. The first figure compares the theoretical result of (9) with the result obtained from a MATLAB simulation using complex Gaussian random noise for $x[n]$ with no windowing, $\mu = 3 - j5$ (so $|\mu|^2 = 34$), and the standard deviations of the real and imaginary parts set to $\sigma_{x_r} = 1$ and $\sigma_{x_i} = 2$; thus $\sigma_x^2 = 5$. An input sequence of $N = 7$ samples was used with a DFT size of $K = 32$. The blue line is the sample mean of $X[k]$ for only 5 trials; the red line is the theoretical result of Eqn. (9). The mean shows the expected asinc structure rotating through the real and imaginary parts of $X[k]$ as the argument of (9) varies. The structure of the mean function is clear, and the agreement with theoretical reasonable, with only 5 trials. The next figure shows the same example for 100 trials; the two curves are now in much closer agreement.





The next figure shows the result for 20 random trials of the same input signal, but with $K = N = 32$, corresponding to the case of Eqn. (10). Notice that the mean of $X[0]$ equals $32(3-j5) = 96-j160$ to a very good approximation, while $X[k]$ approaches zero for all other k, both as predicted.



3 Mean-Square

3.1 Mean-Square of $X_w(\omega)$

The mean-square of $X_w(\omega)$ is (recalling that $w[n]$ is real-valued)

$$\begin{aligned}
\overline{\left|X_w(\omega)\right|^2} &= \overline{\left(\sum_{n=0}^{N-1} w[n]\,x[n]\,e^{-j\omega n}\right)\left(\sum_{m=0}^{N-1} w[m]\,x[m]\,e^{-j\omega m}\right)^*} \\
&= \overline{\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]x[n]x^*[m]\,e^{-j\omega n}\;e^{+j\omega m}} \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\overline{x[n]x^*[m]}\,e^{-j\omega n}\;e^{+j\omega m} \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\overline{\left(\mu+\tilde{x}[n]\right)\left(\mu^*+\tilde{x}^*[m]\right)}\,e^{-j\omega n}\;e^{+j\omega m} \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\overline{\left(\left|\mu\right|^2+\mu^*\tilde{x}[n]+\mu\tilde{x}^*[m]+\tilde{x}[n]\tilde{x}^*[m]\right)}\,e^{-j\omega n}\;e^{+j\omega m}
\end{aligned} \tag{11}$$

Recall also that $\tilde{x}[n]$ is zero mean, and define the autocorrelation function of $\tilde{x}[n]$ as

$$s_{\tilde{x}}[k] = \overline{\tilde{x}[n]\tilde{x}^*[n+k]} \tag{12}$$

Note that $s_{\tilde{x}}[0] = \sigma_{\tilde{x}}^2 = \sigma_x^2$. Using these properties and definitions in (11) gives

$$\begin{aligned}
\overline{\left|X_w(\omega)\right|^2} &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\overline{\left(\left|\mu\right|^2+\mu^*\tilde{x}[n]+\mu\tilde{x}^*[m]+\tilde{x}[n]\tilde{x}^*[m]\right)}\,e^{-j\omega n}\;e^{+j\omega m} \\
&= \left|\mu\right|^2 \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]e^{-j\omega n}\;e^{+j\omega m} + 0 + 0 \\
&\qquad + \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\;e^{+j\omega m} \\
&= \left|\mu\right|^2 \left(\sum_{m=0}^{N-1} w[n]e^{-j\omega n}\right)\left(\sum_{n=0}^{N-1} w[m]e^{+j\omega m}\right) \\
&\qquad + \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\;e^{+j\omega m} \\
&= \left|\mu\right|^2 W(\omega)W^*(\omega) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|W(\alpha)\right|^2 S_{\tilde{x}}(\omega-\alpha)\,d\alpha \\
&= \left|\mu\right|^2 \left|W(\omega)\right|^2 + \frac{1}{2\pi}\left|W(\omega)\right|^2 \otimes_\omega S_{\tilde{x}}(\omega)
\end{aligned} \tag{13}$$

where $S_{\tilde{x}}(\omega)$ is the power spectrum of $\tilde{x}[n]$ (DTFT of $s_{\tilde{x}}[k]$) and the symbol $\otimes_\omega$ denotes circular convolution in frequency on the interval $[-\pi,\pi]$. The derivation of the second term in the second-to-last line is given in Appendix 1 of this memorandum.

If $x[n]$ is white, then $s_{\tilde{x}}[k]=\sigma_x^2\delta[k]$ (and $S_{\tilde{x}}(\omega)=\sigma_x^2$) and (13) reduces to

$$\begin{aligned}\overline{\left|X_w(\omega)\right|^2} &= |\mu|^2\left|W(\omega)\right|^2+\sigma_x^2\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}w[n]w[m]\delta[m-n]e^{-j\omega n}\,e^{+j\omega m}\\ &= |\mu|^2\left|W(\omega)\right|^2+\sigma_x^2\sum_{n=0}^{N-1}\left|w[n]\right|^2 = |\mu|^2\left|W(\omega)\right|^2+\frac{\sigma_{\tilde{x}}^2}{2\pi}\int_{-\pi}^{\pi}\left|W(\omega)\right|^2 d\omega \qquad (14)\\ &= |\mu|^2\left|W(\omega)\right|^2+E_w\sigma_x^2 \quad \left(x[n]\text{ white}\right)\end{aligned}$$

where $E_w \equiv \sum_{n=0}^{N-1}\left|w[n]\right|^2 = (1/2\pi)\int_{-\pi}^{\pi}\left|W(\omega)\right|^2 d\omega$ is the energy in the window sequence. The second version of the result in the second-to-last line follows from the first by Parseval's relation [1].

Finally, in the no-window case, so that $E_w = N$,

$$\overline{\left|X(\omega)\right|^2} = |\mu|^2\left|\frac{\sin[\omega N/2]}{\sin[\omega/2]}\right|^2+N\sigma_x^2 \quad (\text{stationary white input, no window}) \qquad (15)$$

3.2 Mean-Square of $X_w[k]$

The mean square of the DFT is obtained in a manner directly analogous to the DTFT case (see Appendix 1 for some of the details):

$$\begin{aligned}\overline{\left|X_w[k]\right|^2} &= |\mu|^2\left|W[k]\right|^2+\frac{1}{K}\sum_{p=0}^{K-1}W[p]W^*[p]S_{\tilde{x}}[k-p], \quad n=0,\ldots,K-1\\ &= |\mu|^2\left|W[k]\right|^2+\frac{1}{K}\left|W[k]\right|^2\otimes_k S_{\tilde{x}}[k], \quad n=0,\ldots,K-1\end{aligned} \qquad (16)$$

where $\otimes_k$ denotes circular convolution in discrete frequency over the interval $[0,K-1]$. When $x[n]$ is white this becomes

$$\begin{aligned}\overline{\left|X_w[k]\right|^2} &= |\mu|^2\left|W[k]\right|^2+\sigma_x^2\sum_{n=0}^{N-1}w^2[n] = |\mu|^2\left|W[k]\right|^2+\frac{\sigma_x^2}{K}\sum_{k=0}^{K-1}\left|W[k]\right|^2\\ &= |\mu|^2\left|W[k]\right|^2+E_w\sigma_x^2 \quad (\text{stationary white input})\end{aligned} \qquad (17)$$

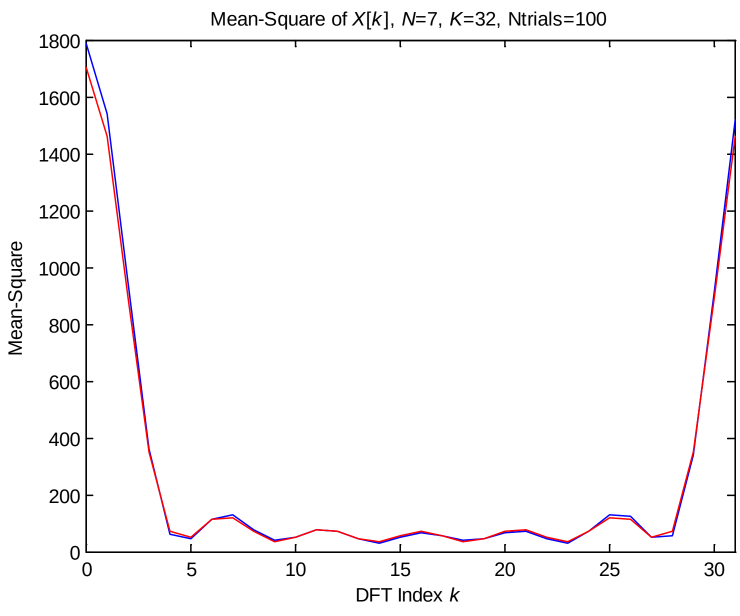
In the no-window case, this reduces further to

$$\overline{|X[k]|^2} = |\mu|^2 \left(\frac{\sin[\pi k N/K]}{\sin[\pi k/K]} \right)^2 + N\sigma_x^2 \quad \text{(stationary white input, no window)} \tag{18}$$

If in addition $K = N$, we have

$$\begin{aligned} \overline{|X[k]|^2} &= N^2 |\mu|^2 \delta[k] + N\sigma_x^2 \\ &= \begin{cases} N^2|\mu|^2 + N\sigma_x^2, & k = 0 \ \text{("DC term")} \\ N\sigma_x^2, & k = 1, \dots, K\text{-}1 \end{cases} \\ &\quad \text{(stationary white input, no window, } K = N\text{)} \end{aligned} \tag{19}$$

The next figure verifies the no window, $K = N$ result of (18) for the same example discussed previously and 100 random trials. Again, the blue line is the sample mean-square while the red line is the theoretical expression of Eqn. (18). The value expected at $k = 0$ is $N^2|\mu|^2 + N\sigma_x^2$ = (49)(34)+(7)(5) = 1701, while the "white noise" level is (7)(5) = 35, consistent with the minimum level of the sidelobes, as seen in the red curve, which is a plot of Eqn. (18). The blue curve is a good match to the theoretical result. If the number of random trials is increased, the blue curve matches the red more closely.



4 Variance

4.1 Variance of $X_w(\omega)$

Since $\sigma_u^2 = \overline{|u|^2} - |\overline{u}|^2 = \overline{|u|^2} - |\mu_u|^2$ for any complex random variable u with mean μ_u, we have for $X_w(\omega)$

$$\begin{aligned}\sigma_{X_w}^2 &= |\mu|^2 |W(\omega)|^2 + \frac{1}{2\pi}\int_{-\pi}^{\pi} |W(\alpha)|^2 S_{\tilde{x}}(\omega-\alpha)\, d\alpha - |\mu W(\omega)|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |W(\alpha)|^2 S_{\tilde{x}}(\omega-\alpha)\, d\alpha \\ &= \frac{1}{2\pi}|W(\omega)|^2 \otimes_\omega S_{\tilde{x}}(\omega) \quad (\text{stationary input})\end{aligned} \tag{20}$$

If $x[n]$ is white, this becomes

$$\sigma_{X_w}^2 = E_w \sigma_x^2 \quad (\text{stationary white input}) \tag{21}$$

and in the no-window case this further reduces to

$$\sigma_X^2 = N\sigma_x^2 \quad (\text{stationary white input, no window}) \tag{22}$$

4.2 Variance of $X_w[k]$

Similarly, the variance of the DFT is again obtained in direct analogy to the DTFT case. The result is

$$\sigma_{X_w}^2 = \frac{1}{K}|W[k]|^2 \otimes_k S_{\tilde{x}}[k], \quad n = 0,\ldots,K-1 \quad (\text{stationary input}) \tag{23}$$

When $x[n]$ is white this becomes

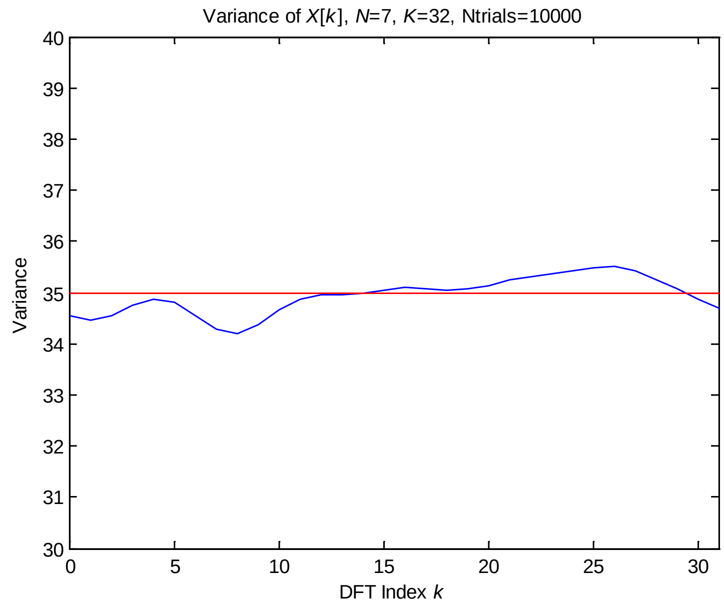
$$\sigma_{X_w}^2 = E_w \sigma_x^2 \quad (\text{stationary white input}) \tag{24}$$

In the no-window case, this reduces further to

$$\sigma_X^2 = N\sigma_x^2 \quad (\text{stationary white input, no window, any window length}) \tag{25}$$

which does not depend on the window length.

Note that, unlike the mean and mean-square, the variance is independent of the DFT index k for any value of the input mean or of the DFT size K. The following figure continues the previous example, this time for 10,000 trials.



5 PDF of $X_w(\omega)$ and $X_w[k]$

We again decompose $x[n]$ into its mean $\mu = \mu_r + j\mu_i$ and a time-varying part $\tilde{x}[n]$ as in Eq. (1). We now also assume that the real and imaginary parts of $\tilde{x}[n]$ are both independent, identically-distributed (iid) Gaussian random processes, with equal variances $\sigma_x^2/2$. The real and imaginary parts of $x[n]$ are therefore Gaussian with equal variances $\sigma_x^2/2$ but non-zero and different means μ_r and μ_i, respectively. To make derivation of the probability density function (PDF) minimally tractable, we will also assume the input noise is stationary and white. We will, however, still use a window $w[n]$. The zero-mean special case is a standard model for such processes as receiver noise and target fluctuations for “many-scatterer” targets, and is therefore a major case of interest.

5.1 Marginal PDFs of Real and Imaginary Parts

We denote the real and imaginary parts of $x[n]$ as $x_r[n]$ and $x_i[n]$, the real and imaginary parts of $X(\omega)$ as $X_r(\omega)$ and $X_i(\omega)$, and of $X[k]$ as $X_r[k]$ and $X_i[k]$. Similar notations apply for $\tilde{x}[n]$ and $\tilde{X}[k]$. When a non-trivial window is present, we will use subscripts *w* or *wr* and *wi* as needed.

It is convenient to consider $\tilde{X}_w(\omega)$ first. The real and imaginary parts of $\tilde{X}_w(\omega)$ are

$$\begin{aligned}\tilde{X}_{wr}(\omega) &= \sum_{n=0}^{N-1} w[n]\left\{\tilde{x}_r[n]\cos(\omega n) + \tilde{x}_i[n]\sin(\omega n)\right\} \\ \tilde{X}_{wi}(\omega) &= \sum_{n=0}^{N-1} w[n]\left\{\tilde{x}_i[n]\cos(\omega n) - \tilde{x}_r[n]\sin(\omega n)\right\}\end{aligned} \tag{26}$$

Since each is a weighted sum of zero-mean Gaussian random variables (r.v.s), it follows that both $\tilde{X}_{wr}(\omega)$ and $\tilde{X}_{wi}(\omega)$ are themselves zero-mean Gaussian r.v.s [3]. However, we cannot yet write the marginal Gaussian PDFs because we do not yet know their variance.

Consider the mean-square of $\tilde{X}_{wr}(\omega)$:

$$\begin{aligned}
\overline{\tilde{X}^2_{wr}(\omega)} &= \overline{\left(\sum_{n=0}^{N-1} w[n]\left\{\tilde{x}_r[n]\cos(\omega n) + \tilde{x}_i[n]\sin(\omega n)\right\}\right)\bullet\ldots} \\
&\qquad \overline{\ldots\bullet\left(\sum_{m=0}^{N-1} w[n]\left\{\tilde{x}_r[m]\cos(\omega m) + \tilde{x}_i[m]\sin(\omega m)\right\}\right)} \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\left\{\overline{\tilde{x}_r[n]\tilde{x}_r[m]}\cos(\omega n)\cos(\omega m) + \ldots\right. \qquad (27) \\
&\qquad \ldots + \overline{\tilde{x}_r[n]\tilde{x}_i[m]}\cos(\omega n)\sin(\omega m) + \overline{\tilde{x}_i[n]\tilde{x}_r[m]}\sin(\omega n)\cos(\omega m) + \ldots \\
&\qquad \left.\ldots + \overline{\tilde{x}_i[n]\tilde{x}_i[m]}\sin(\omega n)\sin(\omega m)\right\}
\end{aligned}$$

Because $\tilde{x}_r[n]$ and $\tilde{x}_i[n]$ are zero mean, white, and i.i.d., (27) reduces to

$$\begin{aligned}
\overline{\tilde{X}^2_{wr}(\omega)} &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\left\{\left(\frac{\sigma_x^2}{2}\right)\delta[n-m]\cos(\omega n)\cos(\omega m) + \ldots\right. \\
&\qquad \ldots + (0)\cos(\omega n)\sin(\omega m) + (0)\sin(\omega n)\cos(\omega m) + \ldots \\
&\qquad \left.\ldots + \left(\frac{\sigma_x^2}{2}\right)\delta[n-m]\sin(\omega n)\sin(\omega m)\right\} \qquad (28) \\
&= \left(\frac{\sigma_x^2}{2}\right)\sum_{n=0}^{N-1} w^2[n]\left\{\cos^2(\omega n) + \sin^2(\omega n)\right\} \\
&= E_w\frac{\sigma_x^2}{2} \qquad (\text{stationary zero-mean white uncorrelated complex input})
\end{aligned}$$

In the no-window case, this becomes

$$\overline{\tilde{X}^2_{wr}(\omega)} = N\frac{\sigma_x^2}{2} \qquad (29)$$

$$(\text{stationary zero-mean white uncorrelated complex input, no window})$$

The same value results for $\overline{\tilde{X}^2_{wi}(\omega)}$. This should not be surprising; since the input noise is zero mean i.i.d. with half the power in the real and imaginary parts, and the mean is zero, the power splits evenly between the real and imaginary parts of the DTFT as well. The total power is just the sum of imaginary and real channel powers, which is again $N\sigma_x^2$.

We can now write the marginal PDFs of the real and imaginary parts of the DFTF, $\tilde{X}_{wr}(\omega)$ and $\tilde{X}_{wi}(\omega)$. They are

$$p_{\tilde{X}_{wr}}\left(\tilde{X}_w\right) = p_{\tilde{X}_{wi}}\left(\tilde{X}_w\right) = \frac{1}{\sqrt{\pi E_w \sigma_x^2}} \exp\left[-\tilde{X}_w^2 \big/ E_w \sigma_x^2\right] \tag{30}$$

$$\left(\text{stationary zero-mean white uncorrelated complex input}\right)$$

Furthermore, because the derivation and result do not depend in any way on the particular value of frequency ω, the marginal PDFs of the real and imaginary parts of the DFT, $\tilde{X}_{wr}[k]$ and $\tilde{X}_{wi}[k]$, are exactly the same.

Now consider $X_w(\omega)$. Since $x[n] = \tilde{x}[n] + \mu$ for $0 \le n \le N-1$, using (6) gives

$$X_w(\omega) = \tilde{X}_w(\omega) + \mu W(\omega) \tag{31}$$

so that

$$\begin{aligned} X_{wr}(\omega) &= \tilde{X}_{wr}(\omega) + \operatorname{Re}\{\mu W(\omega)\} \equiv \tilde{X}_{wr}(\omega) + \gamma_r(\omega) \\ X_{wi}(\omega) &= \tilde{X}_{wi}(\omega) + \operatorname{Re}\{\mu W(\omega)\} \equiv \tilde{X}_{wi}(\omega) + \gamma_i(\omega) \end{aligned} \tag{32}$$

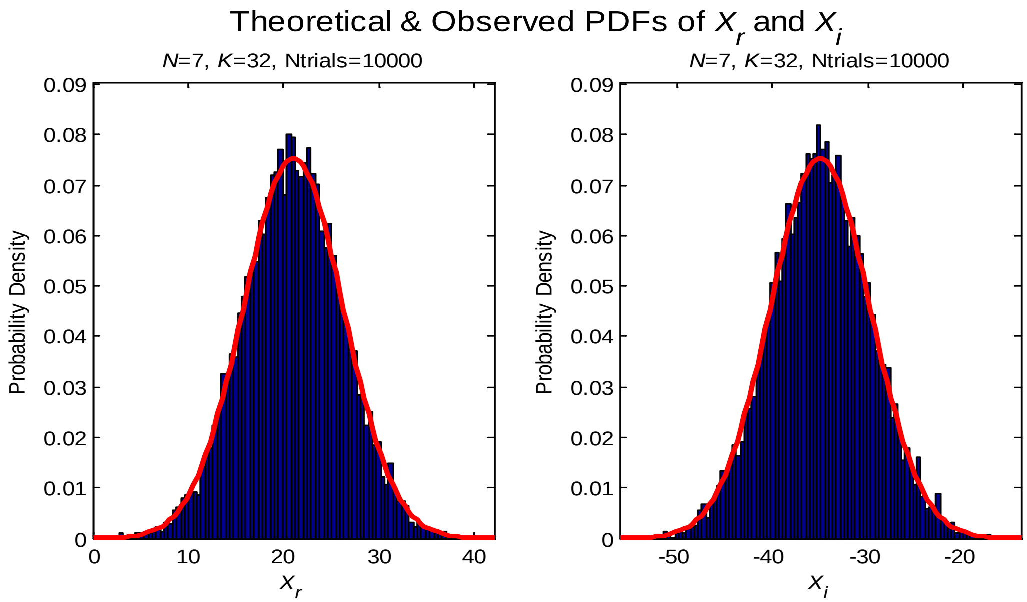
For a given frequency ω, the terms $\gamma_r(\omega)$ and $\gamma_i(\omega)$ are constants. We also define $\gamma(\omega) = \gamma_r(\omega) + j\gamma_i(\omega)$. These shift the means of the Gaussian PDFs of $\tilde{X}_{wr}(\omega)$ and $\tilde{X}_{wi}(\omega)$ but do not affect the variances. Consequently, the marginal PDFs of $X_{wr}(\omega)$ and $X_{wi}(\omega)$ are non-zero mean Gaussians:

$$\begin{aligned} p_{X_{wr}}\left(X_{wr}\right) &= \frac{1}{\sqrt{\pi E_w \sigma_x^2}} \exp\left[-\left(X_{wr} - \gamma_r\right)^2 \big/ E_w \sigma_x^2\right] \\ p_{X_{wi}}\left(X_{wi}\right) &= \frac{1}{\sqrt{\pi E_w \sigma_x^2}} \exp\left[-\left(X_{wi} - \gamma_i\right)^2 \big/ E_w \sigma_x^2\right] \end{aligned} \tag{33}$$

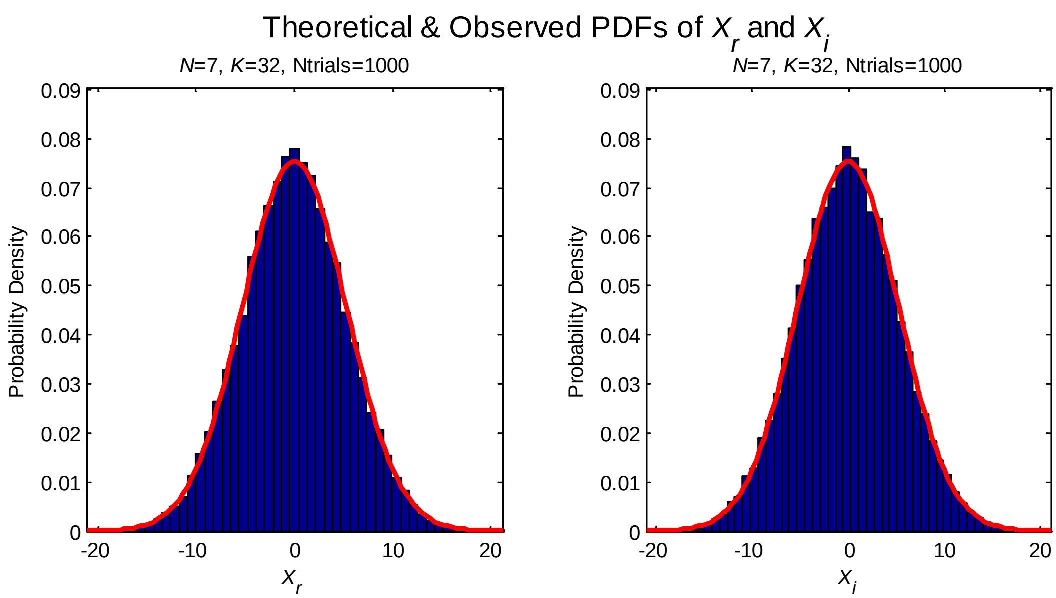
$$\left(\text{stationary non-zero-mean white uncorrelated complex input}\right)$$

In general, the two means are different, so $X_{wr}(\omega)$ and $X_{wi}(\omega)$ are *not* i.i.d. Also, though not shown explicitly for compactness, the mean of the PDF is a function of frequency through $\gamma_r(\omega)$ or $\gamma_i(\omega)$. Finally, since again the particular value of ω does not affect the form of the result, Eqn. (33) also applies to the marginal PDFs of the DFT, with ω evaluated at $2\pi k/K$ for the DFT index k of interest.

The following figures illustrate these results for the same case of N = 7 and K = 32 used earlier. In accordance with the assumptions, the mean of $x[n]$ is also kept at $\mu = 3-j5$, but the variances of the real and imaginary parts are now made equal, specifically a value of 2, so the total variance is $\sigma_x^2 = 4$. The observed PDFs of $X_r[0]$ and $X_i[0]$ for 10,000 trials and the theoretical results of Eqn. (33) are shown in the figure, and agree very well. As the number of trials increases, the fit between the histograms and theoretical results becomes still closer. Note that the means of $X_r[0]$ and $X_i[0]$ at ω or k = 0 are, respectively, $N\mu_r = (7)(3) = 21$ and $N\mu_i = (7)(-5) = -35$.



When the mean of the input is set to zero but all other conditions remain the same, the following magnitude and phase PDFs are observed, now in excellent agreement with Eqn. (30).



5.2 Joint PDF and Correlation of Real and Imaginary Parts

It is not true in general that two marginally Gaussian r.v.s are also jointly Gaussian; see Example 6-3 in [3] for a counter-example. However, it is true ([3], p. 176) that X_{wr} and X_{wi} are jointly Gaussian if any linear combination of them, $\alpha X_{wr} + \beta X_{wi}$, is Gaussian. It is clear that $\alpha \tilde{X}_{wr} + \beta \tilde{X}_{wi}$ is a sum of weighted Gaussian r.v.s and is therefore Gaussian, since $\tilde{X}_{wr}$ and $\tilde{X}_{wi}$ are each sums of weighted Gaussian r.v.s; thus $\tilde{X}_{wr}$ and $\tilde{X}_{wi}$ are jointly Gaussian. It then follows that X_{wr} and X_{wi} are also jointly Gaussian since they merely shift the mean of the distribution away from (0,0). The joint PDF of $X_{wr}(\omega)$ and $X_{wi}(\omega)$ is thus a bivariate Gaussian distribution, and for our particular means and variances is therefore of the form [4]

$$p_{X_{wr}X_{wi}}\left(X_{wr}, X_{wi}\right) = \frac{1}{\pi E_w \sigma_x^2 \sqrt{1-\rho_{ri}^2}} \exp\left\{ -\frac{1}{E_w \sigma_x^2 \left(1-\rho_{ri}^2\right)} \bullet \left[\left(X_{wr}-\gamma_r\right)^2 - 2\rho_{ri}\left(X_{wr}-\gamma_r\right)\left(X_{wi}-\gamma_i\right) + \left(X_{wi}-\gamma_i\right)^2 \right] \right\} \tag{34}$$

where $\rho_{ri} = 2\left(\overline{X_{wr}X_{wi}} - \gamma_r\gamma_i\right) / E_w\sigma_x^2$ is the correlation coefficient of $X_{wr}(\omega)$ and $X_{wi}(\omega)$ ([3], p. 210), again specialized to our particular variances. If $\rho_{ri} = 0$, then $X_{wr}(\omega)$ and $X_{wi}(\omega)$ are uncorrelated and, since also Gaussian, independent. To determine these characteristics, it is convenient to start with $X_w(\omega)$.

An important detail implied by the notation above is that the random variables $X_{wr}(\omega)$ and $X_{wi}(\omega)$ are evaluated at the same frequency in the DTFT. One could consider the joint distribution of $X_{wr}(\omega_1)$ and $X_{wi}(\omega_2)$. However, our primary interest is in deriving the statistics of the magnitude and phase of the DTFT and DFT at a particular frequency, so we will restrict ourselves to the case $\omega_1 = \omega_2 \equiv \omega$.

Remembering that both X_{wr} and X_{wi} are real numbers,

$$\begin{aligned} \overline{X_{wr}X_{wi}} - \gamma_r\gamma_i &= \overline{\left(\gamma_r + \tilde{X}_{wr}\right)\left(\gamma_i + \tilde{X}_{wi}\right)} - \gamma_r\gamma_i \\ &= \gamma_r\gamma_i + \gamma_r\overline{\tilde{X}_{wi}} + \gamma_i\overline{\tilde{X}_{wr}} + \overline{\tilde{X}_{wr}\tilde{X}_{wi}} - \gamma_r\gamma_i \\ &= 0 + 0 + \overline{\tilde{X}_{wr}\tilde{X}_{wi}} \\ &= \overline{\tilde{X}_{wr}\tilde{X}_{wi}} \end{aligned} \tag{35}$$

The last term is

$$\begin{aligned}\overline{\tilde{X}_{wr}(\omega)\tilde{X}_{wi}(\omega)} &= \overline{\left(\sum_{n=0}^{N-1} w[n]\left\{\tilde{x}_r[n]\cos(\omega n)+\tilde{x}_i[n]\sin(\omega n)\right\}\right)\bullet\ldots} \\ &\quad \overline{\ldots\bullet\left(\sum_{m=0}^{N-1} w[m]\left\{\tilde{x}_i[m]\cos(\omega m)-\tilde{x}_r[m]\sin(\omega m)\right\}\right)} \\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\left\{\overline{\tilde{x}_r[n]\tilde{x}_i[m]}\cos(\omega n)\cos(\omega m)+\ldots\right. \\ &\quad \ldots-\overline{\tilde{x}_r[n]\tilde{x}_r[m]}\cos(\omega n)\sin(\omega m)+\ldots \\ &\quad \ldots+\overline{\tilde{x}_i[n]\tilde{x}_i[m]}\sin(\omega n)\cos(\omega m)+\ldots \\ &\quad \left.\ldots-\overline{\tilde{x}_i[n]\tilde{x}_i[m]}\sin(\omega n)\sin(\omega m)\right\}\end{aligned} \tag{36}$$

Again using the zero mean, white, i.i.d properties of $\tilde{x}[n]$, we obtain

$$\begin{aligned}\overline{\tilde{X}_{wr}(\omega)\tilde{X}_{wi}(\omega)} &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]\left\{(0)-\left(\frac{\sigma_x^2}{2}\right)\delta[n-m]\cos(\omega n)\sin(\omega m)+\ldots\right. \\ &\quad \left.\ldots\left(\frac{\sigma_x^2}{2}\right)\delta[n-m]\sin(\omega n)\cos(\omega m)+(0)\right\} \\ &= \left(\frac{\sigma_x^2}{2}\right)\sum_{n=0}^{N-1} w^2[n]\left\{-\cos(\omega n)\sin(\omega m)+\sin(\omega n)\cos(\omega m)\right\} \\ &= 0\end{aligned} \tag{37}$$

The correlation coefficient $\tilde{\rho}_{ri}$ is therefore also equal to zero. Since $\tilde{X}_{wr}$ and $\tilde{X}_{wi}$ are both Gaussian, this in turn implies that they are independent Gaussians ([3], p .211). Furthermore, from (35), (37), and the definition of ρ_{ri} we now see that the correlation coefficient of X_{wr} and X_{wi} is also zero:

$$\rho_{ri} = \frac{2\left(\overline{X_{wr}X_{wi}}-\gamma_r\gamma_i\right)}{E_w\sigma_x^2} = \frac{2\overline{\tilde{X}_{wr}\tilde{X}_{wi}}}{E_w\sigma_x^2} = 0 \tag{38}$$

The joint PDFs of $\tilde{X}_{wr}$ and $\tilde{X}_{wi}$ and of X_{wr} and X_{wi} become

$$\begin{aligned}p_{\tilde{X}_{wr}\tilde{X}_{wi}}\left(\tilde{X}_{wr},\tilde{X}_{wi}\right) &= \frac{1}{\pi E_w\sigma_x^2}\exp\left\{-\frac{1}{E_w\sigma_x^2}\left[\tilde{X}_{wr}^2+\tilde{X}_{wi}^2\right]\right\} \\ &= p_{\tilde{X}_{wr}}\left(\tilde{X}_{wr}\right)p_{\tilde{X}_{wi}}\left(\tilde{X}_{wi}\right)\end{aligned} \tag{39}$$

and

$$p_{X_{wr}X_{wi}}\left(X_{wr},X_{wi}\right)=\frac{1}{\pi E_w\sigma_x^2}\exp\left\{-\frac{1}{E_w\sigma_x^2}\left[\left(X_{wr}-\gamma_r\right)^2+\left(X_{wi}-\gamma_i\right)^2\right]\right\} \\ = p_{X_{wr}}\left(X_{wr}\right)p_{X_{wi}}\left(X_{wi}\right) \tag{40}$$

Thus X_{wr} and X_{wi} are independent. [3]

5.3 PDFs of Magnitude, Magnitude-Squared, and Phase

Finally, since the PDFs of X_r and X_i are non-zero-mean Gaussians with the same variance $E_w\sigma_x^2/2$, the PDFs of $|X[k]|$ and $\phi = \arg\{X[k]\}$ can be found. Using standard results (see pp. 191-192 in [3]), the PDF of $|X[k]|$ is known to be a Rician distribution:

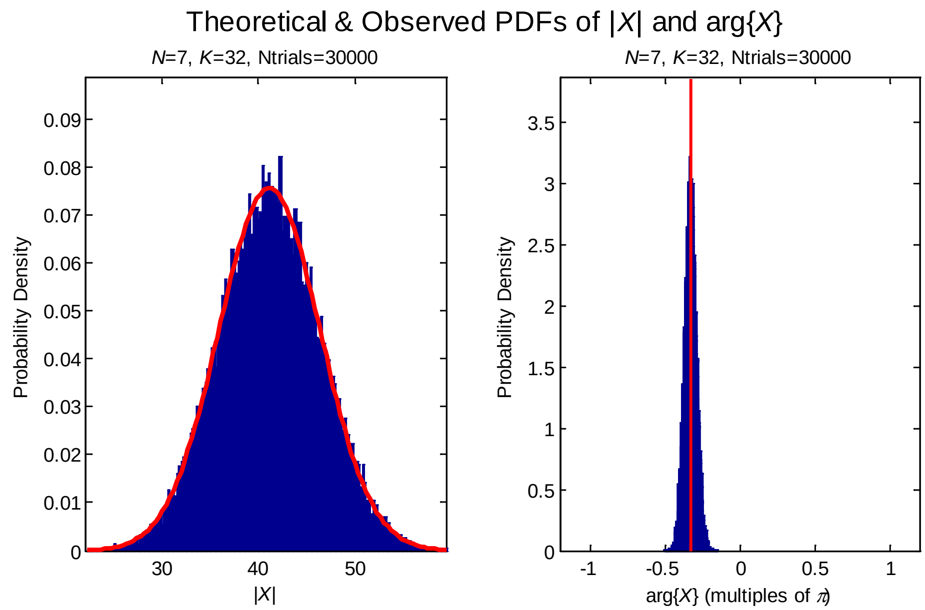
$$p_{|X|}\left(|X|\right)=\begin{cases}\dfrac{2|X|}{E_w\sigma_x^2}\exp\left[-\left(|X|^2+|\gamma|^2\right)\Big/E_w\sigma_x^2\right]I_0\left(\dfrac{2|X||\gamma|}{E_w\sigma_x^2}\right), & |X|\geq 0\\ 0, & \text{otherwise}\end{cases} \tag{41}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind and zero order.

The PDF of the phase ϕ, while conceptually simple, is very lengthy and tedious to compute and does not result in a compact closed form. For example, the joint PDF of Eqn. (40) can be easily converted to polar coordinates using the approach in ([3], pp. 202-203), but the result is lengthy, and must still be integrated over the magnitude to get the marginal PDF for phase. Not surprisingly, however, the phase in the non-zero-mean case clusters around the value arg(γ), and the spread around that value is determined by the variance of X_w.

The following figure shows the observed and predicted PDF of $|X(0)|$ and the observed PDF of $\arg\{X(0)\}$, again for the same case used previously. The mean of $X(0)$ in this case is $\gamma = N\mu = 21-j35$. Note that the PDF of $|X(0)|$ clusters around $|\gamma| = 40.8$, and the phase PDF clusters around $\arg\{\gamma\} = -0.33\pi$ (denoted by the vertical marker line).

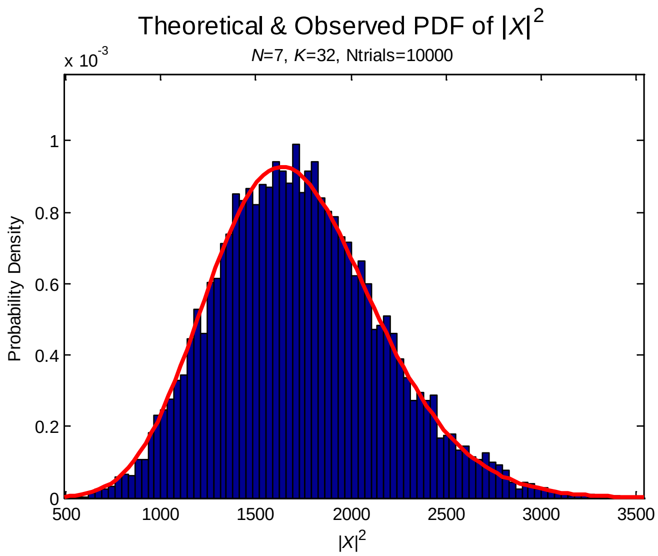
[3] Thanks to Roy Sivley of the MITRE Corporation for correcting an error in a previous version of this memo that resulted in an incorrect conclusion that X_{wr} and X_{wi} are not independent.



The PDF of $Y = |X|^2$ can be obtained from the PDF of $|X|$ using standard results from random processes; see ([3], p. 132). For the magnitude-squared of the DFT or DTFT at a particular frequency, the result is a noncentral chi-square distribution with two degrees of freedom [5]:

$$p_Y(Y) = \begin{cases} \frac{1}{E_w \sigma_x^2} \exp\left[-\left(Y + |\gamma|^2\right) \Big/ E_w \sigma_x^2 \right] I_0\left(\frac{2Y|\gamma|}{E_w \sigma_x^2} \right), & Y \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{42}$$

The next figure shows the very good agreement between this formula and the histogram of the magnitude-squared of the DFT for the same example. Note that the histogram is now centered at about $(40.8)^2 = 1{,}664$.



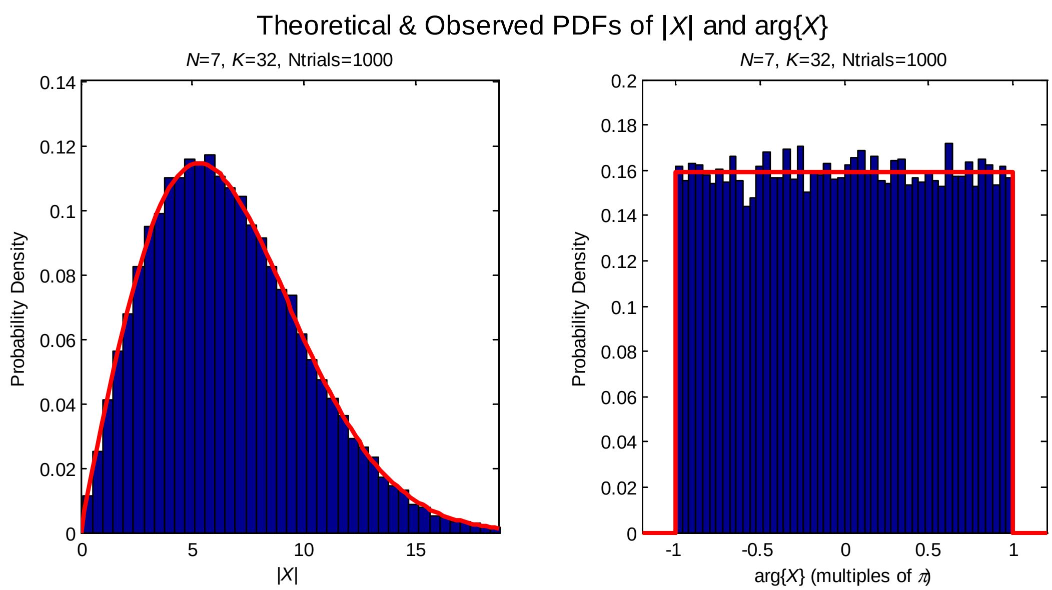
When the mean of the input is zero (*i.e.*, $X_{wr} = \tilde{X}_{wr}$ and $X_{wi} = \tilde{X}_{wi}$, these reduce to the better-known cases (see pp. 190-191 or pp. 202-203 in [3]) of a Rayleigh PDF for the magnitude, exponential PDF for the magnitude-squared, and a uniform PDF for the phase. Specifically,

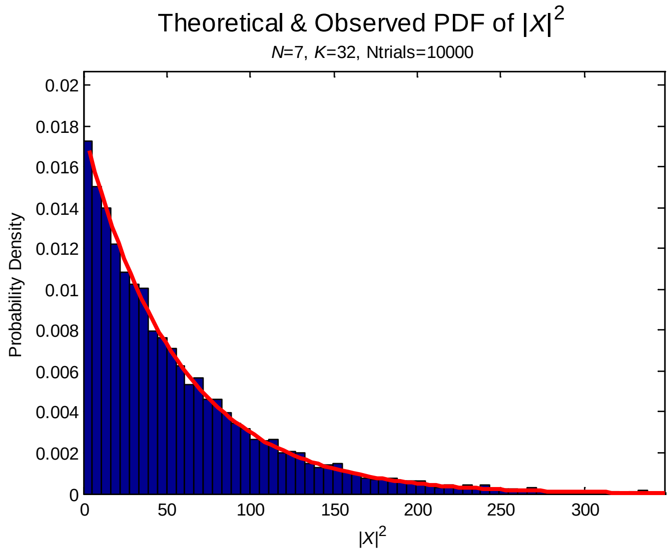
$$p_{|X|}\left(|X|\right) = \begin{cases} \dfrac{2|X|}{E_w \sigma_x^2} \exp\left[-|X|^2 \big/ E_w \sigma_x^2\right], & |X| \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{43}$$

$$p_Y\left(Y\right) = \begin{cases} \dfrac{1}{E_w \sigma_x^2} \exp\left[-Y \big/ E_w \sigma_x^2\right], & Y \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{44}$$

$$p_\phi\left(\phi\right) = \begin{cases} \dfrac{1}{2\pi}, & -\pi \leq \phi \leq \pi \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

For 10,000 trials, the agreement between these formulas and the observed histograms for the same case just shown is very good, as seen in the next two figures.





References

[1] A. V. Oppenheim and R. W. Schafer with J. R. Buck, *Discrete-Time Signal Processing*, 2nd ed. Prentice-Hall, New York, 1999.

[2] M. A. Richards, *Fundamentals of Radar Signal Processing*. McGraw-Hill, New York, 2005.

[3] A. Papoulis and S. U. Pillai, *Probability, Random Variables, and Stochastic Processes*, 4th ed. McGraw-Hill, New York, 2002.

[4] "Bivariate Normal Distribution", Wolfram MathWorld web site, http://mathworld.wolfram.com/BivariateNormalDistribution.html.

[5] S. M. Kay, *Fundamentals of Statistical Signal Processing, Vol. II: Detection Theory*. Prentice-Hall, New York, 1998.

Appendix 1

The goal is to show that

$$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\ e^{+j\omega m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|W(\alpha)\right|^2 S_{\tilde{x}}(\omega-\alpha)\,d\alpha \qquad (46)$$

The approach begins by expressing each function in the summand on the left hand side as the inverse discrete time Fourier transform of the corresponding spectrum, giving

$$\begin{aligned}
&\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\ e^{+j\omega m} \\
&\quad = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\ e^{+j\omega m} \\
&\quad = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \left\{\frac{1}{(2\pi)^3}\int_{-\pi}^{\pi} W(\alpha)e^{j\alpha n}d\alpha \int_{-\pi}^{\pi} W(\beta)e^{+j\beta m}d\beta \int_{-\pi}^{\pi} S_{\tilde{x}}(\chi)e^{j\chi(m-n)}d\chi\right\} e^{-j\omega n}e^{+j\omega m} \\
&\quad = \frac{1}{(2\pi)^3}\iiint d\alpha\, d\beta\, d\chi\, W(\alpha)W(\beta)S_{\tilde{x}}(\chi)\left\{\sum_{n=-\infty}^{\infty} e^{j(\alpha-\chi-\omega)n}\right\}\left\{\sum_{m=-\infty}^{\infty} e^{j(\chi+\beta+\omega)m}\right\}
\end{aligned} \qquad (47)$$

The extension of the summation limits in the second line is possible because the finite length window still limits the summand range.

The last term in brackets is [A1.1]

$$\sum_{m=-\infty}^{\infty} e^{j(\chi+\beta+\omega)m} = 2\pi \sum_{p=-\infty}^{\infty} \delta_D\left(\chi+\beta+\omega+2\pi p\right) \qquad (48)$$

where $\delta_D(\cdot)$ is the Dirac impulse function. Only one impulse occurs within the 2π range of any of the integrands. Choosing to integrate over χ first, we have

$$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\ e^{+j\omega m}$$
$$= \frac{1}{(2\pi)^3}\iint d\alpha\, d\beta W(\alpha)W(\beta)\left\{\sum_{n=-\infty}^{\infty} e^{j(\alpha-\chi-\omega)n}\right\}\bullet$$
$$\bullet\left\{\int_{=\pi}^{\pi} 2\pi\delta_D(\chi+\beta+\omega)S_{\tilde{x}}(\chi)d\chi\right\} \quad (49)$$
$$= \frac{1}{(2\pi)^2}\iint d\alpha\, d\beta W(\alpha)W(\beta)\left\{\sum_{n=-\infty}^{\infty} e^{j(\alpha+\beta)n}\right\}S_{\tilde{x}}(\beta+\omega)$$

where the last step used the sifting property of the impulse function. Evaluating the second summation gives a similar impulse train; integrating next over β results in elimination of the integral and $\beta = -\alpha$. We also note that, for real $w[n]$, $W(-\alpha) = W^*(\alpha)$. Thus we finally obtain the result used in (13):

$$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\ e^{+j\omega m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\alpha)W^*(\alpha)S_{\tilde{x}}(\omega-\alpha)d\alpha$$
$$\equiv \frac{1}{2\pi}\left|W(\omega)\right|^2 \otimes_\omega S_{\tilde{x}}(\omega) \quad (50)$$

The symbol $\otimes_\omega$ thus represents a circular convolution in the frequency domain over the interval $[-\pi,\pi]$.

Exactly the same derivation approach can be used to obtain the similar result for the DFT instead of the DFT. The sequences $w[n]$, $w[m]$, and $s_{\tilde{x}}[m-n]$ are now represented by their inverse discrete Fourier transforms (DFTs) instead of inverse DTFTs; for instance,

$$w[n] = \frac{1}{K}\sum_{k=0}^{K-1} W[k]\exp(+j2\pi nk/K), \quad n = 0,\ldots,K-1 \quad (51)$$

The step analogous to Eqn. (48) will be of the form

$$\sum_{k=0}^{K-1} e^{j2\pi(p+q+k)m/K} = K\sum_{p=-\infty}^{\infty}\delta(p+q+k+rK) \quad (52)$$

For discrete indices p, q, k, and r. The final result will be the DFT analogy to (50),

$$\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} w[n]w[m]s_{\tilde{x}}[m-n]e^{-j\omega n}\ e^{+j\omega m}$$
$$= \frac{1}{K}\sum_{p=0}^{K-1} W[p]W^*[p]S_{\tilde{x}}[k-p], \quad n = 0,\ldots,K-1 \tag{53}$$
$$\equiv \frac{1}{K}\left|W[k]\right|^2 \otimes_k S_{\tilde{x}}[k], \quad n = 0,\ldots,K-1$$

where the symbol $\otimes_k$ denotes a discrete circular convolution in frequency in the interval [0,K–1].

References

[A1.1] M. A. Richards, *Fundamentals of Radar Signal Processing*. McGraw-Hill, New York, 2005.

Appendix 2

The following relations were used in an earlier version of this memo. Though not used for the current results, they are retained here for possible future use. They would most likely be needed if some of the results were expanded to include unequal variances and/or non-zero means in the real and imaginary parts of the input noise, or if the covariance between real and imaginary parts of the spectrum at different frequencies was needed.

Adapting summations #1 and 2 in Section 1.351 (p. 31) of [A2.1] gives

$$\sum_{n=0}^{N-1} \cos^2\left(2\pi nk/K\right) = \frac{N+1}{2} + \frac{\cos\left(N\frac{2\pi k}{K}\right)\sin\left[\left(N-1\right)\frac{2\pi k}{K}\right]}{2\sin\left(\frac{2\pi k}{K}\right)} \qquad (54)$$

$$\sum_{n=0}^{N-1} \sin^2\left(2\pi nk/K\right) = \frac{N-1}{2} - \frac{\cos\left(N\frac{2\pi k}{K}\right)\sin\left[\left(N-1\right)\frac{2\pi k}{K}\right]}{2\sin\left(\frac{2\pi k}{K}\right)} \qquad (55)$$

Using a double-angle formula and adapting summation #1 in Section 1.342 (p. 30) gives

$$\sum_{n=0}^{N-1} \sin\left(\frac{2\pi nk}{K}\right)\cos\left(\frac{2\pi nk}{K}\right) = \frac{\sin\left(2\pi Nk/K\right)\sin\left(2\pi\left(N-1\right)k/K\right)}{2\sin\left(2\pi k/K\right)} \qquad (56)$$

That same summation, along with summation #2 in the same section, gives

$$\sum_{n=0}^{N-1} \sin\left(\frac{2\pi nk}{K}\right) = \frac{\sin\left(\pi Nk/K\right)\sin\left(\pi\left(N-1\right)k/K\right)}{\sin\left(\pi k/K\right)} \qquad (57)$$

$$\begin{aligned}\sum_{n=0}^{N-1} \cos\left(\frac{2\pi nk}{K}\right) &= 1 + \frac{\cos\left(\pi Nk/K\right)\sin\left(\pi\left(N-1\right)k/K\right)}{\sin\left(\pi k/K\right)} \\ &= \frac{\sin\left(\pi Nk/K\right)\cos\left(\pi\left(N-1\right)k/K\right)}{\sin\left(\pi k/K\right)}\end{aligned} \qquad (58)$$

References

[A2.1] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*. Academic Press, New York, 1980.