Vibration Modes and Natural Frequency Veering in Three-Dimensional, Cyclically Symmetric Centrifugal Pendulum Vibration Absorber Systems

This paper investigates the vibration mode structure of three-dimensional, cyclically symmetric centrifugal pendulum vibration absorber (CPVA) systems. The rotor in the system has two translational, one rotational, and two tilting degrees of freedom. The equations of motion for the three-dimensional model, including the rotor tilting, are derived to study the modes analytically and numerically. Only three mode types exist: rotational, translational-tilting, and absorber modes. The rotational and absorber modes have identical properties to those of in-plane models. Only the translational-tilting modes contain rotor tilting. The veering/crossing behavior between the eigenvalue loci is derived analytically. [DOI: 10.1115/1.4025678]

Keywords: centrifugal pendulum vibration absorber, dynamic, vibration, critical speed, stability, rotating system, tilting, veering/crossing

1 Introduction

Centrifugal pendulum vibration absorbers (CPVAs) are used to counteract vibration in rotating machines. They are usually mounted on one end of a rotating element (called the rotor), such as the crankshaft in automotive engines or the main rotor of a helicopter. The distinction between the present model and analysis with prior work is the inclusion of rotor tilting in CPVA systems. The expanded model that includes rotor tilting is suited to the common situation where the absorbers are installed at one end of the rotating shaft. In such cases, the forces and moments that the absorbers exert on the rotor can cause tilting. In other applications, there might be external forces and moments that drive tilting vibration. In both of these situations, the rotor can experience undesirable tilting motion.

CPVAs were used in internal combustion engines as early as 1929 [1]. Since their invention, their function of reducing vibration in rotating systems has been studied widely. Den Hartog [2] analyzed the working principles of CPVAs. The dynamic response and stability of CPVAs are studied by Shaw and his coworkers analytically [3–12]. Albright et al. [13] and Nester et al. [14] conducted experiments to investigate the torsional behavior of CPVAs. These works, however, only considered purely rotational models. Bauchau et al. [15] numerically investigated the translational vibration reduction in Sikorsky UH-60 helicopters using CPVAs. Cronin [16] included both rotor translational and rotational vibrations in his study of shake reduction in four-cylinder engines, while ignoring the rotor bearing stiffness. Shi and Parker [17,18] and Shi et al. [19] considered CPVA systems with rotor rotation and translation as well as multiple absorber groups, but the models were still restricted to in-plane motions.

This paper analyzes the free vibration of three-dimensional CPVA systems (i.e., systems with rotor tilting and translation) that have equally spaced, identical absorbers. The linearized equations of motion are derived using Lagrange's equation. These equations are investigated analytically and numerically to obtain the vibration mode structure of the system. Only three mode types occur: rotational, translational-tilting, and absorber modes. The rotational and absorber modes have identical properties with the in-plane models in Refs. [17] and [18]. The translational-tilting modes differ from the translational modes found in the in-plane models [17,18], and they are discussed in detail. The analysis of rotor-tilting response to external tilting torques requires the understanding of translational-tilting modes, which is crucial for the study of rotor-tilting vibration reduction using CPVAs. The stability of the translational-tilting modes is also investigated.

In Refs. [17] and [18], natural frequency veering behavior was observed in the numerical examples, but no analysis was given. Veering behavior also occurs in three-dimensional systems. This behavior is examined analytically in this paper. The veering/crossing patterns help trace the evolution of eigenvalue loci with varying rotor speed.

2 System Model

The system model used to investigate the three-dimensional, cyclically symmetric CPVA system extends the in-plane model analyzed in Ref. [17] by including a shaft attached to the rotor and aligned with the rotation axis. The shaft is considered as a component of the rotor, as shown in Fig. 1. N identical bifilar absorbers [20–22] are equally spaced on the rotor and mounted at one end of the shaft. The absorbers serve their vibration reduction purpose at a frequency proportional to the rotor rotation speed. The constant of proportionality is defined as the tuning order. The distance between the rotor center and pivot and the radius of the absorber
are denoted by \( k_s \). The torsional tilting stiffness supporting the shaft is \( K_t \). Both \( k_s \) and \( K_t \) represent isotropic supports whose stiffnesses are the same in any translation or tilting direction. Each absorber has mass \( m \). For biar absorbers, there is no absorber moment of inertia and we neglect the inertia of the rollers.

The fixed basis is \( \{ e_1^0, e_2^0, e_3^0 \} \). Figure 1(a) shows additional bases and coordinates associated with the in-plane motion. The reference basis \( \{ e_1^0, e_2^0, e_3^0 \} \) rotates at a constant mean speed \( \Omega \) about \( e_3^0 \). The rotor and the shaft shown in Figs. 1(b) and 1(c) rotate at this constant mean speed \( \Omega \) about \( e_3^0 \) as well. The two translational vibrations along the \( e_1^0 \) and \( e_2^0 \) directions and the rotational vibration of the rotor are denoted by \( x, y, \) and \( \mu \), respectively. Another intermediate reference basis \( \{ e_1^0, e_2^0, e_3^0 \} \) and the shaft-fixed basis \( \{ e_1^0, e_2^0, e_3^0 \} \) are used to depict the rotor three-dimensional motions. The basis \( \{ e_1^0, e_2^0, e_3^0 \} \) fixed on the rotor is defined such that \( e_1^0 \) points from the rotor center to the pivot of the \( i \)th absorber. The fixed angle \( \beta_i \) between \( e_1^0 \) and \( e_1^0 \) describes the pivot position of the \( i \)th absorber. For convenience, \( \beta_i \) is assigned to be zero. The basis \( \{ e_1^0, e_2^0, e_3^0 \} \) is fixed on the \( i \)th absorber. Each absorber has a single arc length degree of freedom along its path that is denoted by \( s_i \). Locally circular (for small deflections) absorber paths, such as circular, cycloidal, and epicycloidal paths [20], are used.

Figures 1(b) and 1(c) depict the geometry of the tilting motions. The distance between the center of mass of the combined rotor-shaft part and the absorber plane is \( L \). The two intermediate reference bases \( \{ e_1^0, e_2^0, e_3^0 \} \) and \( \{ e_1^0, e_2^0, e_3^0 \} \) and the shaft-fixed basis \( \{ e_1^0, e_2^0, e_3^0 \} \) define the two tilting degrees of freedom \( \nu \) and \( \eta \) in Figs. 1(b) and 1(c), respectively.

The angular velocities of the combined rotor-shaft and of the \( i \)th biar pendulum, respectively, are

\[
\begin{align*}
\dot{\omega}^h &= (\Omega + \dot{\mu})E_3 + \dot{\nu}e_1^0 + \dot{\eta}e_2^0 \\
&= \dot{\nu}e_1^0 + \dot{\eta} \cos \nu e_1^0 + (\dot{\eta} \sin \nu + \dot{\mu} + \Omega)e_3^0 \quad (1a) \\
\dot{\omega}' &= (\Omega + \dot{\mu})E_3 + \dot{\nu}'e_1^0 + \dot{\eta}'e_2^0 + \dot{\gamma}'e_3^0 \\
&= \left( \nu \frac{\dot{\gamma}}{r} \sin \eta \right) e_1^0 + \left( \eta \cos \nu - \frac{\dot{\gamma}}{r} \sin \nu \cos \eta \right) e_2^0 \\
&+ \left( \eta \sin \nu + \frac{\dot{\gamma}}{r} \cos \nu \cos \eta + \dot{\mu} + \Omega \right) e_3^0 \quad (1b)
\end{align*}
\]

For an origin at the undeflected position of the center of mass, the position and velocity vectors of the center of mass of the rotor-shaft and the \( i \)th absorber are

\[
\begin{align*}
r &= x e_1^0 + y e_2^0 \quad (2a) \\
\dot{r} &= \left[ \dot{x} - (\Omega + \dot{\mu})y \right] e_1^0 + \left[ \dot{y} + (\Omega + \dot{\mu})x \right] e_2^0 \quad (2b) \\
r_i' &= x e_1^0 + y e_2^0 + L e_3^0 + \dot{e}_1^0 + \dot{e}_2^0 + \dot{e}_3^0 \\
&= \left[ \dot{x} + L \sin \eta + L \cos \eta \cos \beta_i + r \cos \eta \cos \left( \beta_i + \frac{s_i}{r} \right) \right] e_1^0 \\
&+ \left[ \dot{y} - L \sin \nu \cos \eta + (\sin \nu \sin \beta_i + \cos \nu \sin \beta_i) \right] e_2^0 \\
&+ \left[ \dot{\gamma} \sin \nu \sin \eta \cos \left( \beta_i + \frac{s_i}{r} \right) + \cos \nu \sin \eta \cos \beta_i \right] e_3^0 \\
&+ \left( \dot{x} + L \cos \eta + L \sin \nu \sin \beta_i - \cos \nu \sin \eta \cos \left( \beta_i + \frac{s_i}{r} \right) \right) e_3^0 + \left( \dot{y} + \dot{\gamma} \cos \nu \sin \eta \cos \beta_i \right) e_3^0 + \left( \dot{\gamma} \sin \nu \sin \eta \cos \left( \beta_i + \frac{s_i}{r} \right) - \cos \nu \sin \eta \sin \beta_i \right) e_3^0 \\
&+ \left( \dot{\gamma} \sin \nu \sin \eta \sin \beta_i + \cos \nu \sin \eta \cos \beta_i \right) e_3^0 \quad (2c)
\end{align*}
\]

\[
\begin{align*}
r_i' &= a e_1^0 + b e_2^0 + c e_3^0 \quad (2d)
\end{align*}
\]
\[
a = \dot{x} - y(\Omega + \dot{\mu} + L[\dot{\eta}\cos\eta + (\Omega + \dot{\mu})\sin\nu\cos\eta] - I[\dot{\eta}\cos\beta_i\sin\eta + (\Omega + \dot{\mu})(\sin\nu\sin\eta\cos\beta_i + \cos\nu\sin\beta_i)] - J \left[ \frac{\dot{\beta}_i}{r} \cos \beta_i \sin \eta + \dot{\eta} \cos \beta_i + \frac{\dot{\beta}_i}{r} \right] \sin \eta + (\Omega + \dot{\mu}) \times \left[ \sin \nu \sin \eta \cos \left( \frac{\beta_i + \frac{\dot{\beta}_i}{r}}{r} \right) + \cos \nu \sin \left( \frac{\beta_i + \frac{\dot{\beta}_i}{r}}{r} \right) \right] \right) \tag{2e}
\]

\[
b = \dot{y} + x(\Omega + \dot{\mu}) + L \left[ \dot{\eta}\sin\nu\sin\eta + (\Omega + \dot{\mu})\sin\eta - \dot{\nu}\cos\nu\cos\eta \right] + I \left[ \dot{\eta}\cos\beta_i\sin\eta + \dot{\nu}(\cos\nu\sin\eta\cos\beta_i - \sin\nu\sin\beta_i) \right] + (\Omega + \dot{\mu})\cos\eta\cos\beta_i + r \left[ \frac{\dot{\beta}_i}{r} \cos \left( \frac{\beta_i + \frac{\dot{\beta}_i}{r}}{r} \right) \cos \nu \right. \\
+ \sin \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) \sin \nu \sin \eta + \dot{\eta} \cos \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) \sin \nu \cos \eta \\
+ \dot{\nu} \left[ \cos \nu \sin \eta \cos \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) - \sin \nu \sin \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) \right] + (\Omega + \dot{\mu})\cos\eta\cos\beta_i \right) \right) \tag{2f}
\]

\[
c = -L(\dot{\eta}\cos\nu\sin\eta + \dot{\nu}\sin\nu\cos\eta) + I \left[ \dot{\nu}(\sin\nu\sin\eta\cos\beta_i + \cos\nu\sin\beta_i) - \dot{\eta}\cos\beta_i\cos\nu\eta \right] + r \left[ \frac{\dot{\beta}_i}{r} \cos \left( \frac{\beta_i + \frac{\dot{\beta}_i}{r}}{r} \right) \sin \nu + \sin \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) \cos \nu \sin \eta \right] \\
- \dot{\eta} \cos \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) \cos \nu \cos \eta + \dot{\nu} \left[ \sin \nu \sin \eta \cos \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) \right] + \cos \nu \sin \left( \beta_i + \frac{\dot{\beta}_i}{r} \right) \right] \tag{2g}
\]

The angular velocities in Eq. (1) and the linear velocity vectors in Eq. (2) are used to calculate the system kinetic energy from

\[
M_{\nu} = \begin{pmatrix}
m_{\nu} + Nm & 0 & 0 & 0 & 0 & 0 & NmL \\
m_{\nu} + Nm & 0 & -NmL & 0 & 0 & 0 & 0 \\
J_{\nu} + Nm(l + r)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{symmetric} & & & & & & \\
\end{pmatrix} \tag{5a}
\]

\[
M_{\mu} = \begin{pmatrix}
-m\sin\beta_1 & -m\sin\beta_2 & \cdots & -m\sin\beta_N \\
m\cos\beta_1 & m\cos\beta_2 & \cdots & m\cos\beta_N \\
m(l + r) & m(l + r) & \cdots & m(l + r) \\
-ml\cos\beta_1 & -ml\cos\beta_2 & \cdots & -ml\cos\beta_N \\
-ml\sin\beta_1 & -ml\sin\beta_2 & \cdots & -ml\sin\beta_N \\
\end{pmatrix} \tag{5b}
\]

\[
M_{\nu} = \text{diag}(m, m, \ldots, m)^T \tag{5c}
\]

\[
G_{\nu} = \begin{pmatrix}
0 & -m_{\nu} & 0 & NmL & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -NmL^2 \\
\text{skew-symmetric} & & & & \\
\end{pmatrix} \tag{5d}
\]

The system potential energy is \( V = \frac{1}{2}[k_r (x^2 + y^2) + k_s (\nu^2 + \eta^2)] \). Substitution of the kinetic and potential energies of the system into Lagrange’s equations yields the nonlinear equations of motion. Linearization of the equations of motion and application of the trigonometric identities \( \sum_{i=1}^{N} \cos \beta_i = \sum_{i=1}^{N} \sin \beta_i = 0 \) for \( N > 2 \) and \( \sum_{i=1}^{N} \cos^2 \beta_i = N/2 [17] \) yield

\[
M_{\mu}q + \Omega G_{\mu}q + (K_{\theta} - \Omega^2 K_{\Omega})q = F \tag{4a}
\]

\[
q = (x, y, \nu, \eta, s_1, s_2, \ldots, s_N)^T \tag{4b}
\]

\[
M = \begin{pmatrix}
M_{\mu} & M_{\nu} \\
\text{symmetric} & M_{\nu} \\
\end{pmatrix} \tag{4c}
\]

\[
G = 2 \begin{pmatrix}
G_{\nu} & G_{\nu} \tag{4d}
\end{pmatrix}
\]

\[
K_{\theta} = \begin{pmatrix}
K_{\theta} & 0_{N \times N} \\
0_{N \times N} & K_{\theta} \tag{4e}
\end{pmatrix}
\]

\[
K_{\Omega} = \begin{pmatrix}
K_{\nu} & K_{\nu} \tag{4f}
\end{pmatrix}
\]

\[
F = (F_x, F_y, T, T_x, T_y, 0, 0, \ldots, 0)^T \tag{4g}
\]

where \( F_x \) and \( F_y \) are the external rotor forces, \( T \) is the external rotor torque about the rotation axis, and \( T_x \) and \( T_y \) are the external tilting moments applied to the rotor. No external loads act directly on the absorbers. The submatrices in Eq. (4) are
The simplest cases of rotational modal properties are the same as those of rotational modes. Examination of absorber modes analytically shows that, although the two rotor-tilting motions are mathematically coupled with the absorber motions in Eq. (4), the absorber modes do not have rotor tilting, so this coupling is not active for absorber modes. Substitution of the proposed absorber mode eigenvector

\[ \phi = (0, 0, \mu, 0, 0, s, s, \ldots, s)^T \]

into the eigenvalue problem of Eq. (4) results in the same reduced rotational mode eigenvalue problem derived in Ref. [17]. Hence, the modal properties of the rotational modes are identical to those for in-plane systems. There are two rotational modes, and one of them is a rigid-body mode. The phase index of the rotational modes is \( k = 0 \), as shown in Refs. [17] and [18]. The nonrigid-body rotational mode eigenvalue is

\[ \lambda_{1,2} = j \Omega \left[ 1 + \frac{N m (l + r)^2}{J_r} \right] \frac{l}{r} \]

where \( j \) is the imaginary unit. The nonrigid-body rotational mode eigenvalue is purely imaginary and increases linearly with the rotor speed (Fig. 3) [17].

Figures 2(e) and 2(f) display an absorber mode of the system in Table 1. No rotor motion exists in the absorber mode. All the absorber amplitudes are equal, but they are not in phase with each other (absorber modes are complex-valued, which is typical of gyroscopic systems). The absorber phases increase sequentially by \( k \cdot 360 \text{deg}/N \) for each adjacent absorber, where \( k \) is the phase index of the mode [17,18,24] \( k = 2 \) for the mode in Figs. 2(e) and 2(f). Therefore, the absorber mode is a traveling wave mode [23]. As for rotational modes, these properties are identical to those of absorber modes for in-plane systems [17,18].
Fig. 2 Structured vibration modes of three-dimensional CPVA system with six equally spaced, identical absorbers and the system parameters given in Table 1. The horizontal axis labels denote the system degrees of freedom. The modes are normalized such that $\psi^T C \psi = 1$, where $\psi$ and $C$ are defined in a later section.
Refs. [17] and [18]. For the unity tuning case (270 deg for other cases).

The coordinates \( m_r \) value has multiplicity identical to those of in-plane systems. The absorber mode eigenvalues, as in Refs. [17] and [18]. translational-tilting mode eigenvalues is degenerate with the traveling wave modes [23]. The coordinates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotor mass, ( m_\text{r} ) (kg)</td>
<td>11</td>
</tr>
<tr>
<td>Rotor inertia about shaft axis, ( J_r ) (kg-m(^2))</td>
<td>0.2</td>
</tr>
<tr>
<td>Rotor-tilting inertia, ( J_t ) (kg-m(^2))</td>
<td>( 1 \times 10^6 )</td>
</tr>
<tr>
<td>Rotor translational stiffness, ( k_r ) (N/m)</td>
<td>( 1 \times 10^9 )</td>
</tr>
<tr>
<td>Rotor-tilting stiffness, ( K_t ) (N-m)</td>
<td>( 1 \times 10^9 )</td>
</tr>
<tr>
<td>Absorber mass, ( m ) (kg)</td>
<td>0.9</td>
</tr>
<tr>
<td>Distance between the center of mass and rotor, ( L ) (m)</td>
<td>0.5</td>
</tr>
<tr>
<td>Distance between center and pivot, ( l ) (m)</td>
<td>0.04</td>
</tr>
<tr>
<td>Absorber radius, ( r ) (m)</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Fig. 3 Eigenvalues of three-dimensional CPVA system with six equally spaced, identical absorbers and the system parameters given in Table 1 for varying rotor speed. Rotational modes are shown by solid (blue) lines, translational-tilting modes are shown by dashed (red) lines, and absorber modes are shown by dotted (black) lines. The inset figures zoom in on the highlighted regions A, B, C, and D.

Based on the above observations, the proposed translational-tilting mode eigenvector is

\[
\phi = \left( x, jx, 0, v, jv, \alpha \beta_1, \alpha \beta_2, \ldots, \alpha \beta_\text{abs} \right)^T
\]  

Because the absorber phases increase sequentially by \( k' \cdot 360 \text{deg} = 360 \text{deg}/N \), the phase index of this mode is \( k = 1 \). The complex conjugate of this mode has a phase index \( k' = -1 \) that is equivalent to \( k = N - 1 \) for cyclically symmetric CPVA systems with \( N \) absorbers. Therefore, the phase indices of the translational-tilting modes are \( k = 1, N - 1 \).

Substitution of Eq. (9) into the eigenvalue problem of Eq. (4) and use of the trigonometric identities \( \sum_{i=1}^{N} \cos \beta_i = \frac{\sum_{i=1}^{N} \sin \beta_i = 0}{\sum_{i=1}^{N} \sin 2 \beta_i = 0 \text{ and } \sum_{i=1}^{N} \cos^2 \beta_i = \sum_{i=1}^{N} \sin^2 \beta_i = N/2} [17] \) give

\[
\begin{align*}
\lambda^2 & \left[ (m_r + Nm)\dot{x} + jNmLv - \frac{jN}{2}ms \right] \\
& - 2\Omega \left[ f(m_r + Nm)x - Nmlv + \frac{N}{2}ms \right] \\
& + k_r x - \Omega^2 \left[ (m_r + Nm)x + jNmLv - \frac{jN}{2}ms \right] = 0 \\
(10a) \\
\lambda^2 & \left\{ -jNmLx + \left[ J_r + Nml^2 + \frac{N}{2}m(l+r)^2 \right] \nu - \frac{N}{2}mLs \right\} \\
& + 2\Omega \left[ -Nmlx - jNmL\nu + \frac{N}{2}mLs \right] \\
& + K_t \nu - \Omega^2 \left\{ -jNmLx + Nm \left[ L^2 - \frac{(l+r)^2}{2} \right] \nu - \frac{N}{2}mLs \right\} = 0 \\
(10b)
\end{align*}
\]
\[ i^2(x - L \nu + s) + 2j \Omega(x + jL \nu) - \Omega^2 \left( x - L \nu - \frac{l}{r} \right) = 0 \] (10c)

The rotor rotational equation vanishes for cyclically symmetric CPVA systems. The second rotor translation equation (not shown) is linearly dependent on the first one shown in Eq. (10a). The same is true for rotor tilting, so only one of the two equations is given in Eq. (10b). Simplications of the N absorber equations all yield Eq. (10c). Hence, the substitution generates only the three equations in Eq. (10). Writing these into matrix form yields the reduced translational-tilting mode eigenvalue problem as

\[ i^2 \mathbf{M}^{(i)} \phi_i + 2j \Omega \mathbf{G}^{(i)} \phi_i + (\mathbf{K}^{(i)}_\Omega - \Omega^2 \mathbf{K}^{(i)}_\Omega) \phi_i = 0 \] (11a)

\[ \phi_i = (x, \nu, s)^T \] (11b)

\[ \mathbf{M}^{(i)} = \begin{pmatrix} \frac{2m_i}{Nm} + 2 & 2L & -j \\ -2jL & \frac{2\nu}{Nm} + 2L^2 + (l + r)^2 & -L \\ j & -L & 1 \end{pmatrix} \] (11c)

\[ \mathbf{G}^{(i)} = \begin{pmatrix} -2j \frac{m_i}{Nm} + 1 & 2L & -1 \\ -2L & -2jL^2 & jL \\ 1 & jL & 0 \end{pmatrix} \] (11d)

\[ \mathbf{K}^{(i)}_\Omega = \begin{pmatrix} 2k_i \frac{N_i}{Nm} & 0 & 0 \\ 0 & 2K_i \frac{N_i}{Nm} & 0 \\ 0 & 0 & 0 \end{pmatrix} \] (11e)

\[ \mathbf{K}^{(i)}_\Omega = \begin{pmatrix} \frac{2m_i}{Nm} + 2 & 2L & -j \\ -2jL & 2L^2 + (l + r)^2 & -L \\ j & -L & \frac{l}{r} \end{pmatrix} \] (11f)

The eigenvalue problem in Eq. (11) is a 3 \times 3 gyroscopic eigenvalue problem with complex coefficients that gives six eigensolutions for translational-tilting modes, all with distinct natural frequencies. Substitution of the complex conjugate of Eq. (9) into the eigenvalue problem of Eq. (4) yields the complex conjugate of Eq. (11) that gives the complex conjugates of the six eigensolutions of Eq. (11). Hence, six pairs of complex conjugate translational-tilting modes exist in the vibration mode structure. The complete eigenvectors \( \phi \) are constructed from Eq. (9) and the reduced eigenvectors \( \phi_i \) in Eq. (11b).

Solving Eqs. (10a) and (10c) for the relation between the coordinates \( \nu \) and \( s \) yields

\[ \nu = \begin{cases} \frac{i^2 (2m_i + Nm) + 2j \Omega Nm + \Omega^2 \left[ 2(m_i + Nm) - r \right]}{(i^2 - 2j \Omega - \Omega^2) m_i L + k_i L} \\ \frac{k_i \left( i^2 + \Omega^2 \right) r}{(i^2 - 2j \Omega - \Omega^2) \left( i^2 - 2j \Omega - \Omega^2 \right) m_i L + k_i L} \end{cases} \] (12)

Therefore, given that \( i \) is purely imaginary except for unstable situations at high speeds, as discussed later, the coordinate \( x \) is 90 deg or 270 deg out of phase with the coordinates \( \nu \) and \( s \). Hence, the rotor-tilting motion \( \nu \) and the first absorber motion \( s \) are in phase or 180 deg out of phase, while the rotor translation \( x \) is 90 deg or 270 deg out of phase with them (Fig. 2d). Because of these phase relationships, translational-tilting mode eigenvectors can be normalized so that \( \nu \) and \( s \) are real and \( i \) is purely imaginary. Magnitude normalization is separate (and usually such that \( \mathbf{M} \phi = 1 \)).

Table 2 shows the natural frequencies of the system in Table 1 at rotor speed \( \Omega = 2000 \text{ rpm} \) (209.44 rad/s). There are two rotational modes. One is a rigid-body mode with zero natural frequency. There is one absorber mode natural frequency with multiplicity \( N - 3 \). The rotational and absorber mode eigenvalues are the same as for the in-plane model [17]. The difference between the vibration mode structures of the in-plane and three-dimensional models comes from the translational-tilting modes (called translational-tilting modes in in-plane models). Six translational-tilting modes exist for any number of absorbers.

The eigenloci of the example system in Table 1 with six equally spaced, identical absorbers are plotted in Fig. 3. Because all eigensolutions occur as complex conjugate pairs, the eigenloci are symmetric about the rotor speed axis; only the positive imaginary eigenvalues are shown. All eigenloci results match the derived results, as do the eigenvector properties when results such as in Fig. 2 are generated at any speed.

### 3.2 Discussion of the Vibration Mode Structure

From the analytical derivations, there are two rotational modes, six translational-tilting modes, and \( N - 3 \) absorber modes associated with a single degenerate eigenvalue of multiplicity \( N - 3 \). Summing these gives \( N + 5 \) modes. This equals the total degrees of freedom. Hence, there is no possible mode type other than rotational, translational-tilting, and absorber modes. Similar properties exist in three-dimensional helical planetary gear systems [25].

The above analytical and numerical investigations reveal that the tilting motion is coupled only with the rotor translation, resulting in the translational-tilting modes. Why does the tilting motion exist only in translational-tilting modes but not in rotational and absorber modes? This question can be answered by investigating the net forces and torques that the absorbers exert on the rotor.

Because the absorber motions in the rotational, translational-tilting, and absorber modes are exactly the same as for planar CPVA systems [17,18], the absorber forces and torques acting on the rotor are identical to those of planar systems. According to Refs. [17] and [18], the absorbers exert a net torque about the

<table>
<thead>
<tr>
<th>Mode type</th>
<th>( N = 4 )</th>
<th>( N = 5 )</th>
<th>( N = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotational</td>
<td>428.20</td>
<td>430.50</td>
<td>432.79</td>
</tr>
<tr>
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<td>418.25</td>
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<td>418.87</td>
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<td>8482.3</td>
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Table 2 Natural frequencies, \( \omega \) (rad/s), for three-dimensional CPVA systems with 4, 5, and 6 equally spaced, identical absorbers at rotor angular speed \( \Omega = 2000 \text{ rpm} \) (209.44 rad/s). The system parameters are given in Table 1. Multiplicities are shown in parentheses.
rotation axis but no net force on the rotor for rotational modes; they exert only a net force but no net torque (about the rotation axis) on the rotor for translational-tilting modes, and no net absorber force or torque is exerted on the rotor for absorber modes. The net force, which is present only in the translational-tilting modes, is the key quantity. In the planar model, its effect is only to cause translation of the rotor. In the three-dimensional model, however, the net absorber force acting in the plane of the absorbers has an offset \( L \) from the rotor-shaft center of mass. Therefore, this net absorber force generates a tilting torque about the rotor-shaft center of mass and results in rotor tilting. Thus, only the translational-tilting modes involve tilting motion.

The tilting torque in translational-tilting modes can be used to counteract the external tilting torques \( T_x \) and \( T_y \) acting on the rotor and thus reduce rotor-tilting vibrations. This requires derivation of rules for tuning order selection, as done in Ref. [19] for rotor translation and rotation.

4 Critical Speeds and Flutter Instability

Following the terminology common for gyroscopic systems, critical speeds are those speeds where an eigenvalue vanishes. Two nonzero critical speeds exist in Fig. 3 at 7.53 \( \times 10^3 \) rad/s and 32.6 \( \times 10^3 \) rad/s. This differs from the in-plane model, where exactly one nonzero critical speed occurs [17,18]. These critical speeds are associated with the two smallest translational-tilting mode eigenvalues.

Because the rotational and absorber modes of the three-dimensional system are the same as the in-plane system, so are their critical speed properties. The rotational and absorber modes do not have nonzero critical speeds or experience flutter instability [17,18].

To examine translational-tilting mode critical speeds, substitution of \( \lambda = 0 \) into Eq. (11a) yields an eigenvalue problem whose solution gives the critical speeds; this is

\[
(K_n^{(c)} - \Omega_n^{(C)} \mathbf{K}_n^{(A)}) \phi_n^{(e)} = 0 \quad (14)
\]

The characteristic equation of Eq. (14) is

\[
\Omega_n^2 \left( 2N_m n (l+r)^2 - L^2 (2l+r) + N_n^2 m (l+r)^2 (2l+r) \right) \Omega_n^4 + 2N_n n_k \left( L^2 (2l+r) - l(l+r)^2 \right) + 2m_n K_n l + N_m n_k (2l+r) \Omega_n^2 - 4k_n K_n l = 0 \quad (15)
\]

Setting aside \( \Omega_n^2 = 0 \), Eq. (15) yields a quadratic equation for \( \Omega_n^2 \), from which one can show that Eq. (15) has two positive solutions for \( \Omega_n^2 \). Thus, there are exactly two nonzero critical speeds in the system and these are associated with the translational-tilting modes. This matches the numerical result shown in Fig. 3, Substitution of the parameters in Table 1 into Eq. (15) gives two nonzero critical speeds that equal the critical speeds in Fig. 3.

Flutter instability (complex eigenvalues with positive real parts) also occurs in Fig. 3. The imaginary parts of the second and third critical speeds that equal the critical speeds in Fig. 3. These numerical results suggest that eigenvalue loci of the same mode type veer, while the eigenvalue loci of different mode types cross each other. This veering and crossing behavior can also be found in other systems [17,18,26–28].

5 Natural Frequency Loci Veering/Crossing

Natural frequency veering is evident in Fig. 3. The enlarged figures of regions A, B, and C show that the translational-tilting mode loci veer away from each other near 3.16 \( \times 10^3 \) rad/s, 6.51 \( \times 10^3 \) rad/s, and 17.7 \( \times 10^3 \) rad/s. Veering behavior also occurs for the translational-tilting mode loci at other regions in Fig. 3. A translational-tilting mode locus crosses the nonrigid-body rotational and absorber mode loci near 6.00 \( \times 10^3 \) rad/s and 10.0 \( \times 10^3 \) rad/s in Fig. 3. These numerical results suggest that eigenvalue loci of the same mode type veer, while the eigenvalue loci of different mode types cross each other. This veering and crossing behavior can also be found in other systems [17,18,26–28].

Perkins and Mote [26] investigated the veering/crossing criterion using an estimation of the loci curvature in the veering neighborhood of distinct natural frequencies. Lin and Parker [27] used this criterion to derive the veering/crossing conditions for nonrotating planetary gear systems. This curvature criterion is used to derive veering/crossing conditions for the current gyroscopic system.

Gyroscopic eigenvalue problems can be cast in a state space form [29–31]

\[
\begin{align*}
\dot{\mathbf{z}} C \psi + D \psi &= 0 \quad (18a) \\
\psi &= (\lambda, \phi, \psi)^T \quad (18b) \\
C &= \begin{pmatrix} M_{n} & 0 \\ 0 & K_n - \Omega_n^2 \mathbf{K}_n \end{pmatrix} \\
D &= \begin{pmatrix} \Omega \mathbf{G} & K_n - \Omega_n^2 \mathbf{K}_n \\ - (K_n - \Omega_n^2 \mathbf{K}_n) \end{pmatrix} \quad (18c)
\end{align*}
\]

An eigenvector \( \psi_n \) is normalized so that \( \psi_n^T C \psi_n = 1 \) for \( a = \pm 1, \pm 2, \ldots, \pm (N+S) \), where negative subscripts denote the complex conjugate eigensolutions of the corresponding positive subscript eigensolutions. Thus, \( \psi_n^T D \psi_n = -\lambda_n \). The eigenvectors of \( C \) [Eq. (18)] satisfy the orthogonality conditions \( \psi_n^T C \psi_n = \psi_n^T D \psi_n = 0 \) for \( a \neq b \).

Applying a small perturbation to the rotor speed such that \( \Omega \rightarrow \Omega + \epsilon \) gives

\[
C + \epsilon C' + \epsilon^2 C'' + \epsilon^3 D + \epsilon^4 D'' \quad (19a)
\]
Fig. 4 Translational-tilting mode eigenvalues of three three-dimensional CPVA systems for varying rotor speed. Each system has six equally spaced, identical absorbers, and the system parameters are given in Table 1, except for the tuning order \( n \) and distance between the rotor-shaft center of mass and rotor plane \( L \). The translational-tilting mode eigenvalue loci of the systems with (a) \( n = 2, L = 0.5 \text{ m} \), (b) \( n = 2, L = 0.8 \text{ m} \), and (c) \( n = 0.8, L = 0.5 \text{ m} \) are shown by dashed (red), solid (blue), and dotted (black) lines, respectively.

First, we consider that the unperturbed eigenvalue \( \lambda_a \) is distinct, that is, either a rotational or translational-tilting mode eigenvalue. The possible eigenvalue degeneracy at an isolated speed where this eigenvalue may cross another eigenvalue does not need special consideration, because we can prescribe that the unperturbed speed is slightly away from such a possible crossing. For distinct \( \lambda_a \), Eq. (23) holds for all \( b \neq a \). For \( b = a \), substitution of Eq. (19d) into \( \psi_a^T C \psi_a = 1 \) and use of the normalization conditions \( \psi_a^T C \psi_a = 1 \) and \( \psi_a^T D \psi_a = -\lambda_a \) give \( c_a = \psi_a^T C \psi_a / 2 \).

Extending the perturbation method to second order, the curve of the eigenvalue locus represented by \( \lambda_a^{(2)} \) [27] is derived as

\[
\lambda_a^{(2)} = -\lambda_a \psi_a^T C \psi_a + \psi_a^T D \psi_a - \lambda_a \psi_a^T C \psi_a + \psi_a^T D \psi_a
\]

The last three terms of Eq. (24) involve \( \psi_a^T \) and, hence, the unperturbed eigenvector expansion in Eq. (22). When two eigenvalue loci are close (i.e., an eigenvalue \( \lambda_b \) is close to the unperturbed eigenvalue \( \lambda_a \)), the coefficient \( c_b \) is large because of the small denominator in Eq. (23). Thus, the eigenvalue locus curvature in Eq. (24) can be approximated by only the terms that contain \( c_b \).

This approximate curvature is defined as the coupling factor \( \chi_{ab} \) of the eigenvalue loci of modes \( a \) and \( b \). The analogous quantity determined from perturbation of the \( n \)th eigensolution \((\lambda_a, \psi_a)\) is \( \chi_{an} \).

Substitution of Eqs. (19), (22), and (23) into Eq. (24) yields the coupling factors of modes \( a \) and \( b \) as

\[
\chi_{ab} = \frac{4|\tilde{\lambda}_b \tilde{\psi}_a^T G \psi_a - \Omega \tilde{\psi}_a^T K \psi_a|^2}{\lambda_b - \lambda_a}
\]

and

\[
\chi_{bn} = \frac{4|\tilde{\lambda}_b \tilde{\psi}_n^T G \psi_n - \Omega \tilde{\psi}_n^T K \psi_n|^2}{\lambda_b - \lambda_n}
\]

For \( \lambda_b \approx \lambda_a \), which has already been used to obtain Eq. (25), \( \chi_{bn} = -\chi_{ab} \).

Equations (25a) and (25b) are valid for distinct rotational and translational-tilting modes.
Equation (25) needs modification to calculate the coupling factors that involve degenerate absorber modes. When mode \( a \) is a degenerate absorber mode and mode \( b \) is a distinct rotational or translational-tilting mode, the coupling factor \( \chi_{ab} \) between them is

\[
\chi_{ab} = \sum_{j} \frac{4|\mu_{ab}|^2}{\lambda_{ab} - \lambda_{a}} \Omega_{ab} \Phi_{j}^{T} \mathbf{K}_{j} \Phi_{j} \left| \frac{\lambda_{ab} - \lambda_{a}}{\lambda_{ab} - \lambda_{b}} \right|^2
\]  

(26)

where the summation index \( j \) covers all absorber modes associated with the degenerate absorber mode eigenvalue \( \lambda_{a} \). The coupling factor \( \chi_{ab} \) is still given by Eq. (25b). The calculation of the coupling factors between two degenerate absorber modes needs more modifications. These coupling factors are of no interest, however, as discussed below.

Equations (25) and (26) are used to investigate the veering/crossing behavior between the eigenvalue loci. If the coupling factor is zero, the curvature of the eigenvalue locus associated with mode \( b \) is independent of mode \( a \) and \( b \) cross with no interaction. Otherwise, the two eigenvalue loci veer. The separation of the two eigenvalue loci \( |\lambda_{ab} - \lambda_{a}| \) affects their concavities significantly. Large coupling factors result from large eigenvalue loci curvatures and yield sharp eigenvalue loci veering.

As shown in Eqs. (25) and (26), the rotor bearing and shaft-tilting stiffnesses (which occur only in \( \mathbf{K}_{a} \)) do not explicitly appear in the coupling factors, and neither does the mass matrix. The coupling factors are determined only by the gyroscopic and centripetal acceleration terms.

Five cases of veering/crossing behavior are considered: (i) two rotational mode eigenvalue loci, (ii) two translational-tilting mode eigenvalue loci, (iii) rotational and translational-tilting mode eigenvalue loci, (iv) rotational and absorber mode eigenvalue loci, and (v) translational-tilting and absorber mode eigenvalue loci. The veering/crossing behavior between two absorber mode eigenvalue loci is not necessary, because there is only one absorber mode eigenvalue locus. Even when multiple absorber modes exist, as is the case with multiple groups of absorbers, the absorber mode eigenvalues are proportional to the rotor speed and never approach each other as the rotor speed increases.

5.1 Two Rotational Modes. To examine the veering/crossing behavior between two rotational modes, two rotational modes of the form in Eq. (6) are substituted into Eq. (25). This yields

\[
\chi_{ab} = \frac{4|\mu_{ab}|^2}{\lambda_{ab} - \lambda_{a}} \left| \frac{N \Omega_{ab}}{r} l^{(b)}(x) s^{(a)}(x) \right|^2 = -\chi_{ba}
\]

(27)

where the superscripts denote from which mode the corresponding degrees of freedom come. The coupling factors vanish for nonzero speeds if and only if either \( s^{(a)} = 0 \) or \( g^{(b)} = 0 \). For nonrigid-body rotational modes, this is never the case, so the eigenvalues veer. Based on the modal properties of rotational modes, \( \lambda_{a} \) and \( \lambda_{b} \) are purely imaginary, and thus so are the coupling factors in Eq. (27). This is appropriate because \( \chi_{ab} \) is associated with \( \chi_{a}^{b} \) and \( \lambda_{b} \) is imaginary.

Only the kinetic energy of the absorber motions affects the coupling factor in Eq. (27); rotor rotational motion is absent in Eq. (27). Larger absorber kinetic energy means larger coupling factor and sharper veering between the two rotational modes.

For the system in Fig. 3, no veering/crossing behavior between rotational mode loci exists. This is because there is only one group of absorbers in the system, which results in only one nonrigid-body rotational mode. For systems with multiple groups of absorbers [18], multiple nonzero rotational mode natural frequencies exist. The veering phenomenon between two nonrigid-body rotational mode loci can be found in those systems and follows the results given above.

5.2 Two Translational-Tilting Modes. Substitution of two translational-tilting modes that have the form in Eq. (9) into Eq. (25) yields

\[
\chi_{ab} = \frac{4|\mu_{ab}|^2}{\lambda_{ab} - \lambda_{a}} \left| \Omega_{ab} \left( l^{(b)}(x) s^{(a)}(x) + \frac{1}{r} g^{(b)}(x) x^{(a)} \right) \right|^2
\]

(28)

where \( \Omega_{ab} \) is given in Eq. (25b). The calculation of the coupling factors between two degenerate absorber modes needs more modifications. These coupling factors are of no interest, however, as discussed below.

Equations (25) and (26) are used to investigate the veering/crossing behavior between the eigenvalue loci. If the coupling factor is zero, the curvature of the eigenvalue locus associated with mode \( b \) is independent of mode \( a \) and \( b \) cross with no interaction. Otherwise, the two eigenvalue loci veer. The separation of the two eigenvalue loci \( |\lambda_{ab} - \lambda_{a}| \) affects their concavities significantly. Large coupling factors result from large eigenvalue loci curvatures and yield sharp eigenvalue loci veering.

As shown in Eqs. (25) and (26), the rotor bearing and shaft-tilting stiffnesses (which occur only in \( \mathbf{K}_{a} \)) do not explicitly appear in the coupling factors, and neither does the mass matrix. The coupling factors are determined only by the gyroscopic and centripetal acceleration terms.

Five cases of veering/crossing behavior are considered: (i) two rotational mode eigenvalue loci, (ii) two translational-tilting mode eigenvalue loci, (iii) rotational and translational-tilting mode eigenvalue loci, (iv) rotational and absorber mode eigenvalue loci, and (v) translational-tilting and absorber mode eigenvalue loci. The veering/crossing behavior between two absorber mode eigenvalue loci is not necessary, because there is only one absorber mode eigenvalue locus. Even when multiple absorber modes exist, as is the case with multiple groups of absorbers, the absorber mode eigenvalues are proportional to the rotor speed and never approach each other as the rotor speed increases.

5.3 Rotational and Translational-Tilting Modes. If the two modes \( a \) and \( b \) in Eq. (25) represent a rotational mode and a translational-tilting mode, the coupling factor \( \chi_{ab} \) is

\[
\chi_{ab} = \frac{4|\mu_{ab}|^2}{\lambda_{ab} - \lambda_{a}} \left| \frac{m \Omega_{ab} l^{(b)}(x) s^{(a)}(x)}{r} \sum_{i=1}^{N} e^{-j\beta_{i}} \right|^2 = 0
\]

(29)

Similarly, \( \chi_{ba} = 0 \). Hence, a rotational mode locus and a translational-tilting mode locus cross when they approach each other.

5.4 Rotational and Absorber Modes. Substitution of the rotational and absorber mode eigenvectors in Eqs. (6) and (8) into Eqs. (25) and (26) gives

\[
\chi_{ab} = \sum_{k=1}^{N-2} \frac{4|\mu_{ab}|^2}{\lambda_{ab} - \lambda_{a}} \left| \frac{m \Omega_{ab} l^{(b)}(x) s^{(a)}(x)}{r} \sum_{i=1}^{N} e^{-j\beta_{i}} \right|^2 = 0
\]

(30a)

\[
\chi_{ba} = \sum_{k=1}^{N-2} \frac{4|\mu_{ab}|^2}{\lambda_{ba} - \lambda_{b}} \left| \frac{m \Omega_{ba} l^{(b)}(x) s^{(a)}(x)}{r} \sum_{i=1}^{N} e^{-j\beta_{i}} \right|^2 = 0
\]

(30b)

where \( k \) is the phase index of the absorber mode and is an integer between 2 and \( N - 2 \). Therefore, rotational and absorber mode loci do not couple, and they will cross if they approach each other.

Because both the nonrigid-body rotational mode and the absorber mode natural frequencies are proportional to the rotor speed and because there is only one of each, they do not approach each other as speed increases (Fig. 3). When multiple groups of absorbers are present, however, there are more than one of these mode types [18]. In that case, the eigenvalues cross as derived above.

5.5 Translational-Tilting and Absorber Modes. The coupling factors between a translational-tilting mode locus and an absorber mode locus are

\[
\chi_{ab} = \sum_{k=1}^{N-2} \frac{4|\mu_{ab}|^2}{\lambda_{ab} - \lambda_{a}} \left| \frac{m \Omega_{ab} l^{(b)}(x) s^{(a)}(x)}{r} \sum_{i=1}^{N} e^{-j(k-1)\beta_{i}} \right|^2 = 0
\]

(31a)

\[
\chi_{ba} = \sum_{k=1}^{N-2} \frac{4|\mu_{ab}|^2}{\lambda_{ba} - \lambda_{b}} \left| \frac{m \Omega_{ba} l^{(b)}(x) s^{(a)}(x)}{r} \sum_{i=1}^{N} e^{-j(k-1)\beta_{i}} \right|^2 = 0
\]

(31b)

where \( k = 2, 3, \ldots, N - 2 \) is the phase index of the absorber mode. Thus, translational-tilting and absorber mode loci are uncoupled and cross when they approach each other.
6 Conclusions

After deriving the equations of motion, this paper investigates the vibration mode structure of three-dimensional, cyclically symmetric CPVA systems. Three mode types are identified (rotational, translational-tilting, and absorber modes), and these are the only possible mode types. The rotational and absorber modes have the same modal properties as for planar CPVA systems. Rotor-tilting motion exists only in the translational-tilting modes, which are complex-valued for this gyroscopic system. The properties of these modes, including all of the phase relationships between the various degrees of freedom of the complex-valued modes, are analytically derived.

The system has exactly two nonzero critical speeds that are associated with the translational-tilting modes. Flutter instability occurs only for translational-tilting modes and at high rotor speeds. The flutter instability conditions derived for in-plane systems are not valid for three-dimensional systems. Systems with small tuning order or large distance between the rotor-shaft center of mass and rotor plane can experience flutter instability more easily.

The veering/crossing behavior between any two eigenvalue loci is derived analytically to show that two eigenvalues of the same type always veer while two of different type always cross. The veering sharpness is determined only by the gyroscopic and centrifugal acceleration terms of the system.

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We thank Steve Shaw of Michigan State University for his help on discovering an error in the equations of motion. We also thank Chris Cooley for the discussion on the calculation of the second-order perturbation of the eigenvalues.

References