

November 3, 2015



# Math 3012 - Applied Combinatorics Lecture 20

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# Reminder

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**Test 3** Tuesday, November 24, 2015. Details on material for which you will be responsible were sent by email after class the preceding Thursday. Again, I ask all of you to study hard. Experience shows that the closing portion of this course has most content. The concepts and techniques will have lasting value.

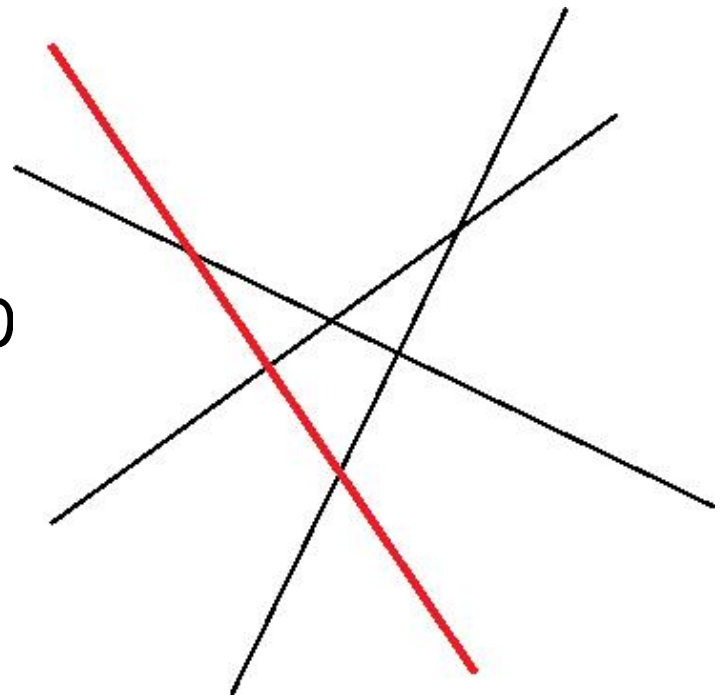
# Review of Recurrence Equations (1)

**Problem** Let  $r(n)$  denote the number of regions determined by  $n$  lines that intersect in general position.

**Solution**

$$r(1) = 2$$

$$r(n + 1) = r(n) + n + 1 \quad \text{when } n \geq 0$$



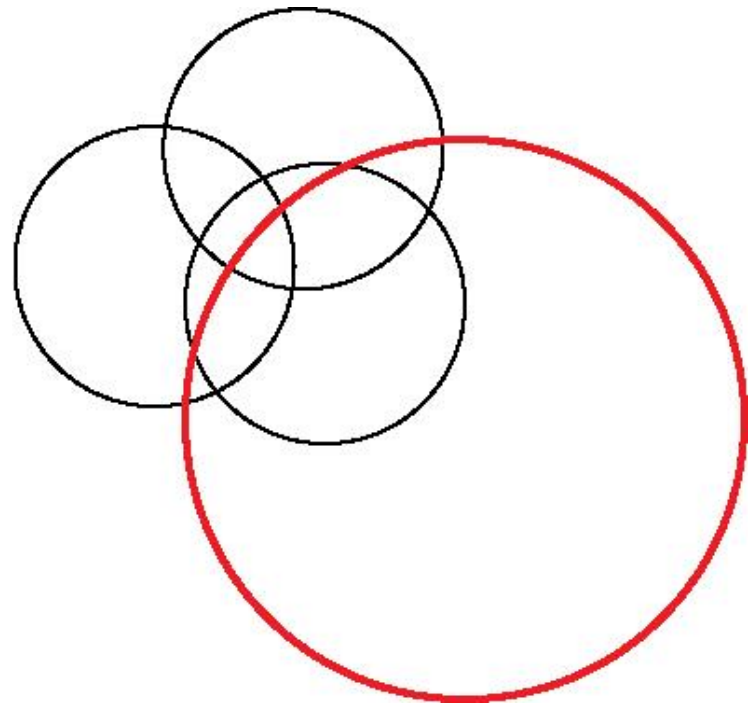
# Review of Recurrence Equations (2)

**Problem** Let  $s(n)$  denote the number of regions determined by  $n$  circles that intersect in general position.

**Solution**

$$s(1) = 2$$

$$s(n+1) = s(n) + 2n \quad \text{when } n \geq 0.$$



# Review of Recurrence Equations (3)

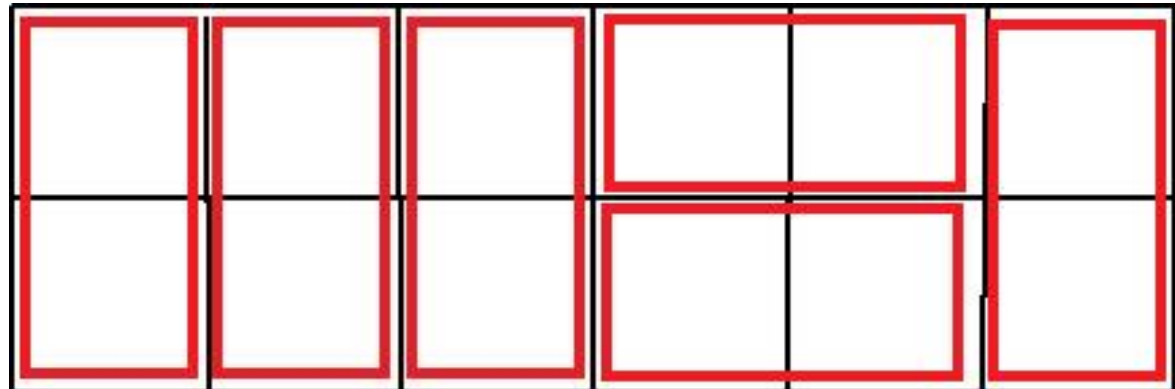
**Problem** Let  $t(n)$  denote the number of ways to tile a  $2 \times n$  grid with dominoes of size  $1 \times 2$  and  $2 \times 1$ .

**Solution**

$$t(1) = 1$$

$$t(2) = 2$$

$$t(n+2) = t(n+1) + t(n) \quad \text{when } n \geq 0.$$



# Review of Recurrence Equations (4)

**Problem** Let  $u(n)$  denote the number of ternary sequences that do not contain 01 in consecutive positions.

**Solution**

$$u(1) = 3$$

$$u(2) = 8$$

$$u(n+2) = 3u(n+1) - u(n)$$

# Review of Recurrence Equations (5)

**Summary** The recurrence equations in the last four examples are:

$$r(n+1) - r(n) = n+1$$

$$s(n+1) - s(n) = 2n$$

$$t(n+2) - t(n+1) - t(n) = 0$$

$$u(n+2) - 3u(n+1) + u(n) = 0$$

# Developing a General Framework (1)

**Observation** We consider the family  $V$  of all functions which map the set  $Z$  of all integers (positive, negative and zero) to the set  $C$  of complex numbers. This is a more general framework than we first studied, but as will become clear, we need this additional structure to make the form of general solutions relatively easy to obtain.

**Note** Each of the four examples presented above have involved functions with range and domain being the set  $N$  of positive integers, so  $V$  is a more general setup.



# Developing a General Framework (2)

**Fact** The family  $\mathcal{V}$  is an infinite dimensional vector space over the field  $\mathcal{C}$  of complex numbers, with  $(f + g)(n) = f(n) + g(n)$  and  $(a f)(n) = a(f(n))$ .

**Note** Students should spot the “operator overloading” in these two equations, even when one of the two operators (multiplication) is indicated simply by adjacent symbols, one a scalar and the other a vector.

**Note** The “zero” of  $\mathcal{V}$  is the constant function which maps all integers to the “zero” in  $\mathcal{C}$ .

# Developing a General Framework (3)

**Observation** We will first focus on homogeneous linear recurrence equations. These have the following form:

$$a_0f(n+d) + a_1f(n+d-1) + a_2f(n+d-2) + \dots \\ + a_{d-1}f(n+1) + a_df(n) = 0$$

**Note** The coefficients  $a_0, a_1, a_2, \dots, a_d$  are complex numbers. Without loss of generality  $a_0 \neq 0$ . For the time being, we will also assume that  $a_d \neq 0$ .

# Developing a General Framework (4)

**Example** A homogeneous equation:

$$(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = 0$$

**Example** A non-homogeneous equation:

$$(2+3i)g(n+3) - (8-7i)g(n+2) + 42g(n+1) - (5i)g(n) = (2-i)(3+i)^n + 12n^3$$

**Remark** In order to fully understand the homogeneous case, we will need to discuss the non-homogeneous case concurrently.

# Developing a General Framework (5)

**Alternate Notation** We define the advancement operator  $A$  on the vector space  $V$  by the rule  $A f(n) = f(n+1)$ . Note that  $A^2 f(n) = f(n+2)$ ,  $A^3 f(n) = f(n+3)$ , etc. So our linear homogeneous equation

$$a_0 f(n+d) + a_1 f(n+d-1) + a_2 f(n+d-2) + \dots \\ + a_{d-1} f(n+1) + a_d f(n) = 0$$

can then be rewritten as:

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \dots + a_{d-1} A + a_d) f(n) = 0.$$

**Remark** The "polynomial form" is significant!

# Developing a General Framework (6)

**Theorem** The set  $S$  of all solutions to a homogeneous linear recurrence equation of the form:

$$(a_0A^d + a_1A^{d-1} + a_2A^{d-2} + \dots + a_{d-1}A + a_d) f_n = 0$$

is a  $d$ -dimensional subspace of  $V$  provided both  $a_0$  and  $a_d$  are non-zero

**Conclusion** The solution space can be specified entirely just by providing a basis for the subspace  $S$ .

# The Case $d = 1$

**Theorem** Let  $a_0$  and  $a_1$  be non-zero complex numbers, and set  $r = (-a_1/a_0)$ . Then the solution space  $S$  of the advancement operator equation  $(a_0A + a_1)f(n) = 0$  is a 1-dimensional subspace of  $V$  and the function  $r^n$  is a basis, i.e., every solution is of the form  $f(n) = c_1 r^n$  where  $c_1$  is a constant.

**Proof** Let  $f$  be any solution to  $(a_0A + a_1)f(n) = 0$ , and let  $c_1 = f(0)$ . We show that  $f(n) = c_1 r^n$  for all integers  $n$ . We first show that  $f(n) = c_1 r^n$  when  $n \geq 0$ . We do this by induction on  $n$ .

# The Case $d = 1$ (Part 2)

**Base Case** The base case is  $n = 0$ , where the left hand side is  $f(0) = c_1$ , and the right hand side is  $c_1 r^0$ . But since  $r \neq 0$ , the right hand side is  $c_1 \cdot 1 = c_1$ . So the statement holds when  $n = 0$ .

**The Inductive Step** Now assume that  $f(k) = c_1 r^k$  for some  $k \geq 0$ . Since  $(a_0 A + a_1)f(n) = 0$  for all integers  $n$ , we know that:

$$(a_0 A + a_1)f(k) = 0$$

$$(a_0 f(k+1) + a_1 f(k)) = 0$$

$$f(k+1) = (-a_1/a_0)c_1 r^k$$

$$f(k+1) = r c_1 r^k$$

$$f(k+1) = c_1 r^{k+1}$$

# The Case $d = 1$ (Part 3)

**Negative Integers** It remains only to show that  $f(n) = c_1 r^n$  for all integers  $n \leq 0$ . This is equivalent to showing that  $f(-n) = c_1 r^{-n}$  for all  $n \geq 0$ . This is done by induction and the argument is a trivial modification of what we have just done.

**Conclusion** We have verified the assertion that the solution space to the homogeneous equation

$$(a_0 A^d + a_1 A^{d-1} + a_2 A^{d-2} + \dots + a_{d-1} A + a_d) f(n) = 0$$

with  $a_0$  and  $a_d$  non-zero is a  $d$ -dimensional subspace of  $V$  when  $d = 1$ .



# Towards the General Case (1)

**Exercise** Note that

$$A^2 + 2A - 35 = (A + 7)(A - 5)$$

**Example** The functions  $(-7)^n$  and  $5^n$  are solutions to the equation:

$$(A^2 + 2A - 35) f(n) = 0.$$

**Observation** If  $r \neq 0$  and  $r$  is a root of the advancement operator polynomial, then  $r^n$  is a solution.

# Towards the General Case (2)

**Exercise** Show that

$$A^2 + (-12 + i)A + 41 - i = (A - 5 - 2i)(A - 7 + 3i)$$

**Example** The functions  $(5 + 2i)^n$  and  $(7 - 3i)^n$  are solutions to the equation:

$$(A^2 + (-12 + i)A + 41 - i) f(n) = 0.$$

**Observation** If  $r \neq 0$  and  $r$  is a root of the advancement operator polynomial, then  $r^n$  is a solution.

# Towards the General Case (3)

**Example** Note that  $A^2 - 10A + 25 = (A - 5)^2$ .

Also note that the functions  $5^n$  and  $n5^n$  are solutions to the equation:

$$(A - 5)^2 f(n) = 0.$$

**Observation** If  $r \neq 0$  and  $r$  is a root of multiplicity 2, then  $r^n$  and  $n r^n$  are solutions.

# Towards the General Case (4)

**Example** The functions  $(5 - 2i)^n$  and  $n(5 - 2i)^n$  are solutions to the equation:

$$(A - 5 + 2i)^2 f(n) = 0.$$

**Observation** If  $r \neq 0$  and  $r$  is a root of multiplicity 2, then  $r^n$  and  $n r^n$  are solutions.

# Towards the General Case (5)

**Lemma** If  $p(A)$  is a polynomial in the advancement operator  $A$ ,  $r \neq 0$  and  $r$  is a root of multiplicity  $m$ , then each of the following functions is a solution of the equation:  $p(A) f(n) = 0$

$$r^n \quad n r^n \quad n^2 r^n \quad n^3 r^n \quad n^4 r^n \quad \dots \quad n^{m-1} r^n$$

**Proof** We will outline the proof in Thursday's lecture.

# Towards the General Case (6)

**Example** The general solution to

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2) f(n) = 0$$

is:

$$\begin{aligned} f(n) = & c_1 3^n + c_2 n 3^n + c_3 n^2 3^n + c_4 n^3 3^n \\ & + c_5 (7 - 2i)^n + c_6 n (7 - 2i)^n + c_7 n^2 (7 - 2i)^n \\ & + c_8 (-5 + 8i)^n + c_9 n (-5 + 8i)^n \end{aligned}$$

# Towards the General Case (7)

**Example** The solution space to:

$$((A - 3)^4(A - 7 + 2i)^3(A + 5 - 8i)^2) f(n) = 0$$

is a 9-dimensional subspace of  $V$  and the following functions are a basis:

$$\begin{array}{cccc} 3^n & n3^n & n^23^n & n^33^n \\ (7 - 2i)^n & n(7 - 2i)^n & n^2(7 - 2i)^n & \\ (-5 + 8i)^n & n(-5 + 8i)^n & & \end{array}$$

# Analogies with Partial Fractions

**Example** Given a proper rational function  $p(x)/q(x)$  whose denominator polynomial  $q(x)$  can be factored as

$$q(x) = (x - 3)^3 x^2(x^2 + 2x + 9)$$

there are constants  $c_1, c_2, c_3, c_4, c_5, c_6$  and  $c_7$  so that

$$\begin{aligned} p(x)/q(x) = & c_1/(x - 3) + c_2/(x - 3)^2 + c_3/(x - 3)^3 \\ & + c_4/x + c_5/x^2 \\ & + c_6/(x^2 + 2x + 9) + c_7x/(x^2 + 2x + 9) \end{aligned}$$



# Analogies with Differential Equations

**Example** Let  $D$  be the differential operator, i.e.,  $Df$  is the derivative of  $f$ . Then the solution to the equation:

$$(D - 3)^3 D^2(D^2 + 16) f = 0$$

has the form:

$$\begin{aligned} f(x) = & c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} \\ & + c_4 + c_5 x \\ & + c_6 e^{4i} + c_7 e^{-4i} \end{aligned}$$

where  $c_6$  and  $c_7$  are complex conjugates.

# The Non-Homogeneous Case

**Theorem** Let  $p(A) f = g$  be a non-homogeneous equation. If  $h_0$  is any solution to this equation, then the general solution is  $h_0 + f$  where  $f$  is a solution to the associated homogeneous equation  $p(A) f = 0$ .

**Note** The proof of this theorem is relatively straightforward.

**Terminology** The function  $h_0$  is referred to as a particular solution to  $p(A) f = g$ .

# The Non-Homogeneous Case (2)

**Example** For the non-homogeneous equation  $(A - 3) f(n) = 8 (5)^n$ , the function  $h_0 = 4 \cdot 5^n$  is a particular solution. Accordingly, the general solution has the form:

$$f(n) = c_1 3^n + 4 \cdot 5^n$$