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Math 3012 – Applied Combinatorics Lecture 9

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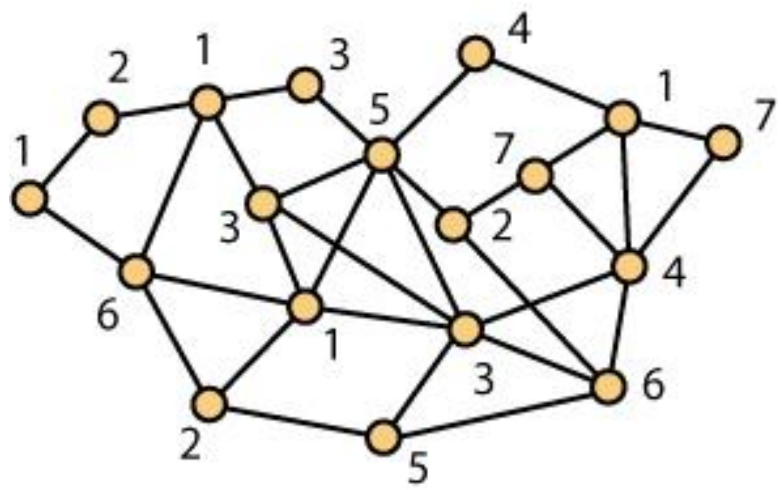
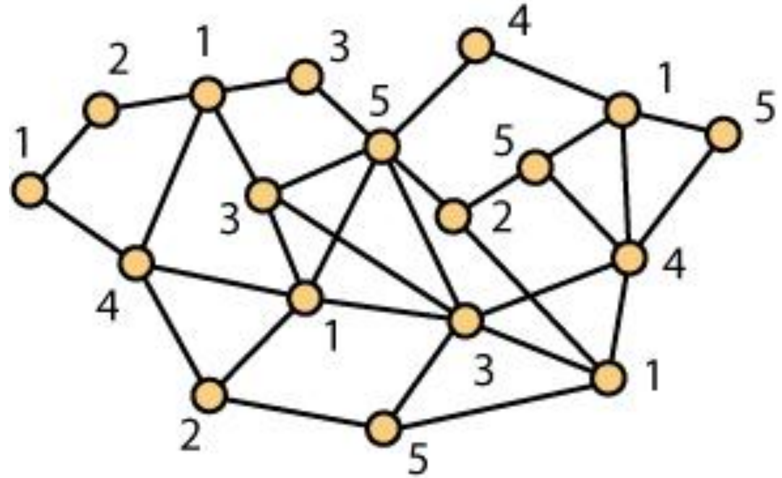
Chromatic Number

Definition A t-coloring of a graph G is an assignment of integers (colors) from $\{1, 2, ..., t\}$ to the vertices of G so that adjacent vertices are assigned distinct colors. We show a 7-coloring of the graph below.



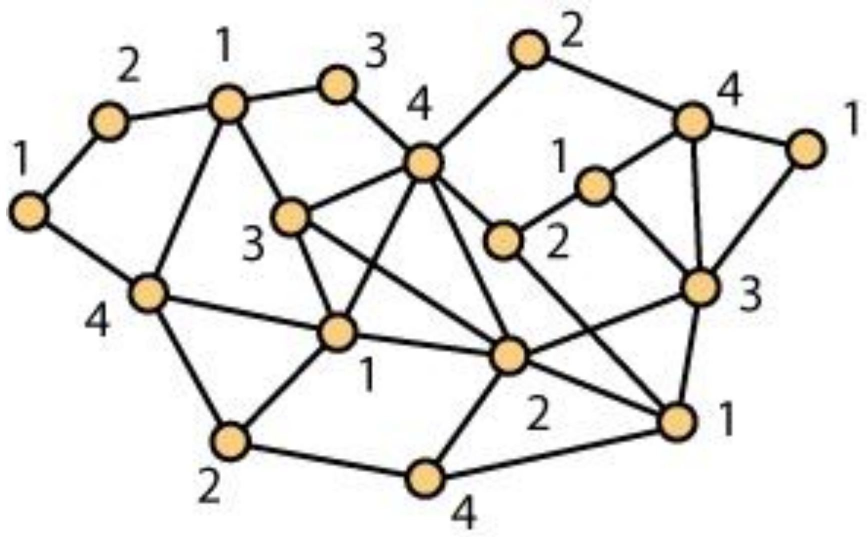
Chromatic Number (2)

Optimization Problems Given a graph G, what is the least t so that G has a t-coloring? This integer is called the chromatic number of G and is denoted $\chi(G)$. The coloring below is the same graph but now we illustrate a 5-coloring, so $\chi(G) \leq 5$.



Chromatic Number (3)

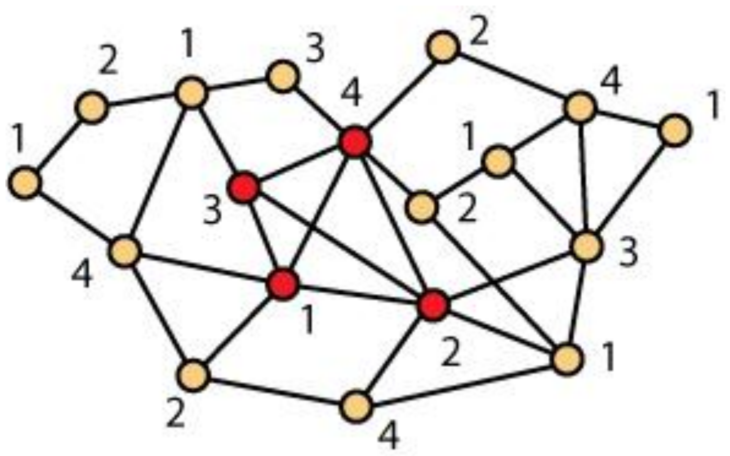
Optimization Problems The coloring below is the same graph but now we illustrate a 4-coloring, so $\chi(G) \leq 4$.



Maximum Clique Size

Definition Given a graph G, the maximum clique size of G, denoted $\omega(G)$, is the largest integer k for which G contains a clique (complete subgraph) of size k.

Trivial Lower Bound $\chi(G) \geq \omega(G)$ so in this case, we know $\chi(G) = \omega(G) = 4$.



Maximum Clique Size

Observation When $n \geq 2$, the odd cycle C_{2n+1} satisfies $\chi(C_{2n+1}) = 3$ and $\omega(C_{2n+1}) = 2$ so the inequality

$$\chi(G) \geq \omega(G)$$

need not be tight. In the remainder of this lecture, we explore this inequality in greater depth, as it is of interest to understand conditions that make this inequality tight, and it is of interest to understand how badly it can fail.

The Inequality Can Fail Arbitrarily

Note In today's class, we will given three different explanations for the following result.

Theorem For every $t \geq 3$, there is a graph G with $\chi(G) = t$ and $\omega(G) = 2$.

Note A clique of size 3 is also called a triangle. Graphs with $\omega(G) \leq 2$ are said to be triangle-free. So the fact can be rephrased as asserting that there are triangle-free graphs with arbitrarily large chromatic number.

A Construction Using the Pigeon-Hole Principle

Basic Idea Proceed by induction. When $t = 3$, take G as the odd cycle C_5. Now suppose that for some $t \geq 3$, we have a triangle-free graph G with $\chi(G) = t$. Here's how we build a new triangle-free graph whose chromatic number is $t + 1$. Suppose G has m vertices labelled $x_1, x_2, \ldots, x_m$.

Start with a "large" independent set Y. For each m-element subset $\{y_1, y_2, \ldots, y_m\}$ of Y, attach a copy of G with x_i adjacent to y_i for each $i = 1, 2, \ldots, m$. This works if Y has size at least $t(m - 1) + 1$ by the Pigeon-Hole principle.

The Mycielski Construction

Basic Idea Proceed by induction. When $t = 3$, take G as the odd cycle C_5. Now suppose that for some $t \geq 3$, we have a triangle-free graph G with $\chi(G) = t$. Here's how we build a new triangle-free graph whose chromatic number is $t + 1$.

Start with a copy of G. Then add an independent set Y containing a "mate" y_x for every vertex x of G. The mate y_x has exactly the same neighbors in G as does x.

Then add one new vertex x_0 which is adjacent to every vertex in Y but to none of the vertices in G.

Shift Graphs

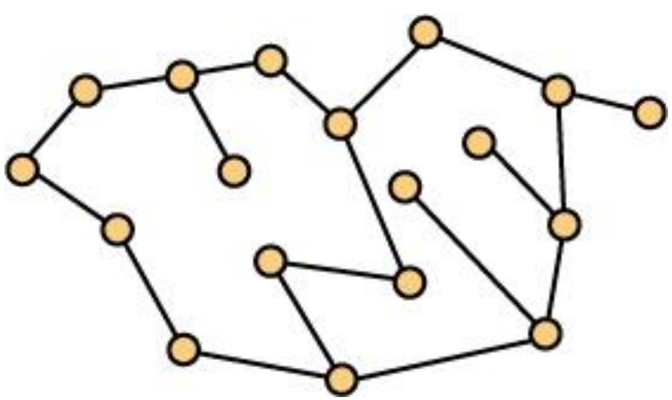
Definition When $n \geq 2$, the shift graph S_n has $C(n, 2)$ vertices and these are the 2-element subsets of $\{1, 2, \ldots, n\}$. For each 3-element subset $\{i, j, k\}$ of $\{1, 2, \ldots, n\}$, with $i < j < k$, the vertex $\{i, j\}$ is adjacent to the vertex $\{j, k\}$ in S_n.

Theorem For every $n \geq 2$, the chromatic number of the shift graph S_n is the least positive integer t so that $2^t \geq n$.

The Girth of a Graph

Definition A graph containing no cycles is called a forest. In a forest, every component is a tree. So a tree is a forest. We say that the girth of a forest is infinite.

Definition When G is not a forest, we define the girth of G as the size of the smallest cycle in G. The graph shown below has girth 8.



Chromatic Number and Girth

Observation The three constructions we have given for triangle-free graphs with large chromatic number produce graphs with small girth. Although the proof is a bit beyond our scope in this course, here is a historically very important result in applications of probability to combinatorics.

Theorem (Erdős, '59) For every pair (g, t) of positive integers with $g, t \geq 3$, there is a graph G with girth g and chromatic number t.