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Math 3012 – Applied Combinatorics Lecture 6

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Math Induction Exercise

Exercise Show that $n^2 > 5n + 13$ when $n \geq 7$.

Math Induction Exercise (2)

Exercise Show that $n^2 > 5n + 13$ when $n \geq 7$.

Attempt at Solution Base Case: $7^2 = 49 > 5 \cdot 7 + 13 = 48$. This works!

Math Induction Exercise (3)

Exercise Show that $n^2 > 5n + 13$ when $n \geq 7$.

Attempt at Solution Base Case: $7^2 = 49 > 5 \cdot 7 + 13 = 48$. This works!

Inductive Step Assume $k^2 > 5k + 13$ for some $k \geq 7$.
Then

$$(k+1)^2 = k^2 + 2k + 1$$
$$> (5k + 13) + (2k + 1)$$
$$= (5k + 5) + (2k + 9)$$

But I need to show that

$$(k+1)^2 > 5(k+1) + 13 = (5k + 5) + 13$$

So I need $2k + 9 \geq 13$. Is this true?

Math Induction Exercise (4)

Exercise If $n \geq 2$, then $2n + 9 \geq 13$

Proof If $n \geq 2$, then $2n \geq 4$, so that $2n + 9 \geq 4 + 9 = 13$.

Exercise Show that $n^2 > 5n + 13$ when $n \geq 7$.

Base Case $7^2 = 49 > 5 \cdot 7 + 13 = 48$. Check!

Inductive Step Assume $k^2 > 5k + 13$ for some $k \geq 7$.

Then
$$
\begin{aligned}
(k + 1)^2 &= k^2 + 2k + 1 \\
&> (5k + 13) + (2k + 1) \\
&= (5k + 5) + (2k + 9) \\
&\geq 5(k + 1) + 13
\end{aligned}
$$

QED

A Much Stronger Result – Calculus!!

Exercise Show that $5n + 13 = o(n^2)$.

Proof Let $\varepsilon > 0$. Then set n_0 be the least positive integer so that $n_0 > 10/\varepsilon$ and $(n_0)^2 > 26/\varepsilon$.
It follows that if $n \geq n_0$, then

$$(5n + 13)/n^2 = 5/n + 13/n^2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

QED

Alternative Forms of Induction

Strategy 1 To argue by contradiction, if a statement S_n is not true for all $n \geq 1$, there is a least positive integer for which it fails.

Strategy 2 To prove that a statement S_n holds for all $n \geq 1$, it is enough to do the following two steps:

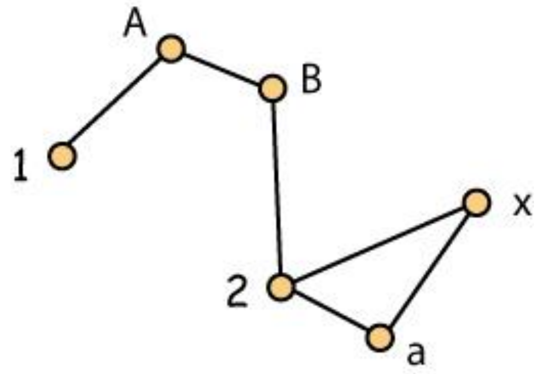
Base Step Verify that the statement S_1 is valid.

Strong Inductive Step Assume that for some $k \geq 1$, the statement S_m is valid for all m with $1 \leq m \leq k$. Then show that statement S_{k+1} is valid.

An Introduction to Graph Theory

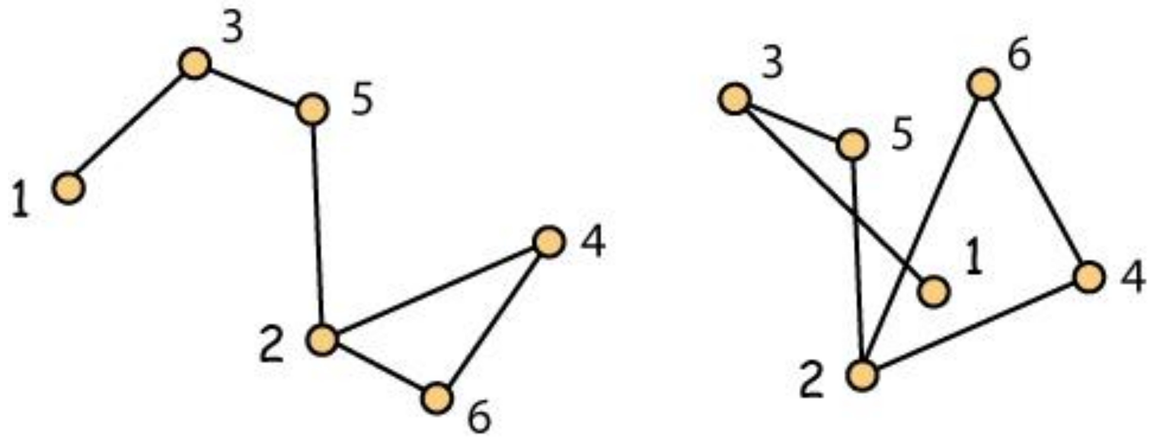
Definition A graph G is a pair (V, E) where V is a finite set and E is a set of 2-element subsets of V. The set V is called the vertex set of G and the set E is called the edge set of G.

Example G = (V, E) where V = {1, 2, A, x, B, a} and E = { {1, A}, {2, x}, {x, a}, {A, B}, {B, 2}, {2, a}}.



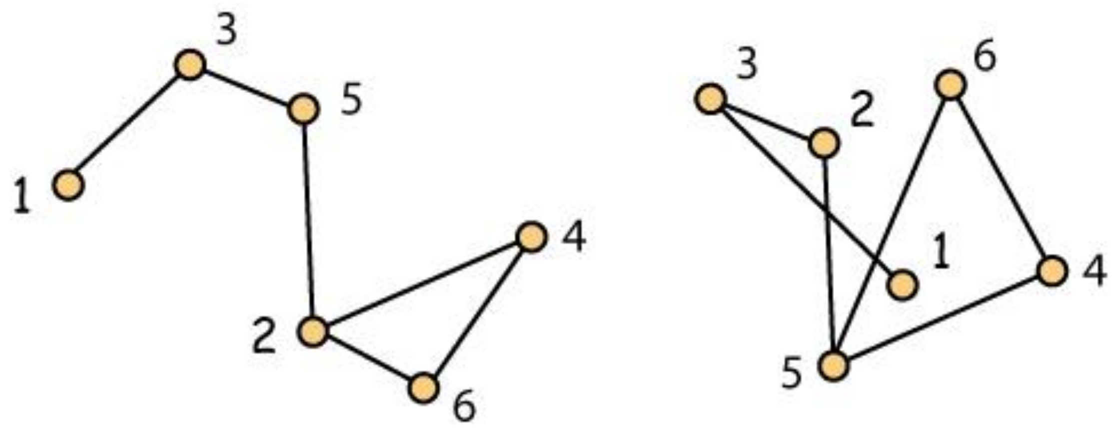
Its All About Adjacency

Comment We show below two drawings of the same graph whose vertex set is {1, 2, 3, 4, 5, 6}.



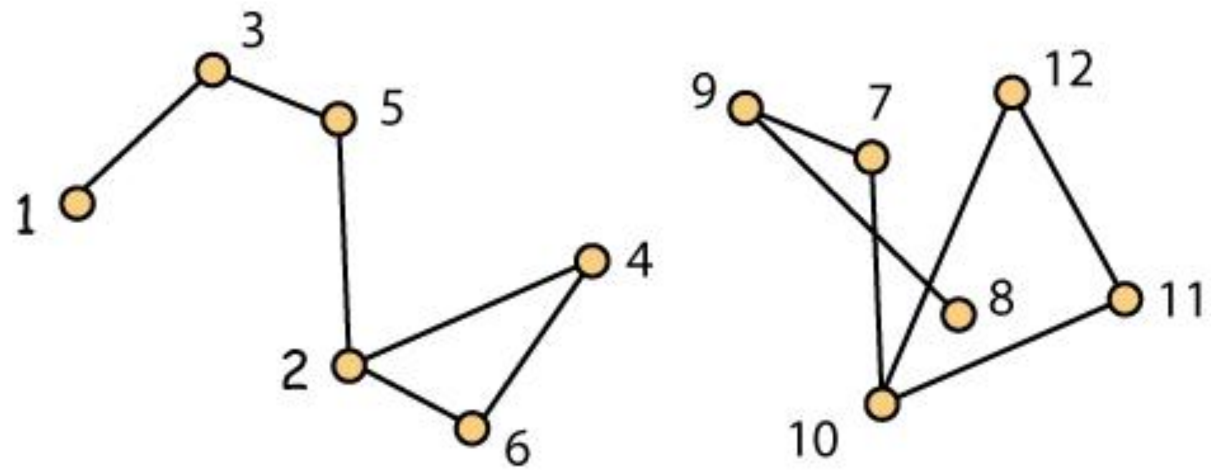
Its All About Adjacency (2)

Comment We show below two drawings of graphs, each having vertex set {1, 2, 3, 4, 5, 6}, but now they represent different graphs.



Its All About Adjacency (3)

Question Is this a drawing of one graph whose vertex set is {1, 2, 3, …, 12} or do we have drawings of two graphs, one with vertex set {1, 2, 3, 4, 5, 6} and the other {7, 8, 9, 10, 11, 12}?



Answer Depends on the meaning of V in the pair (V, E).

Notation and Terminology

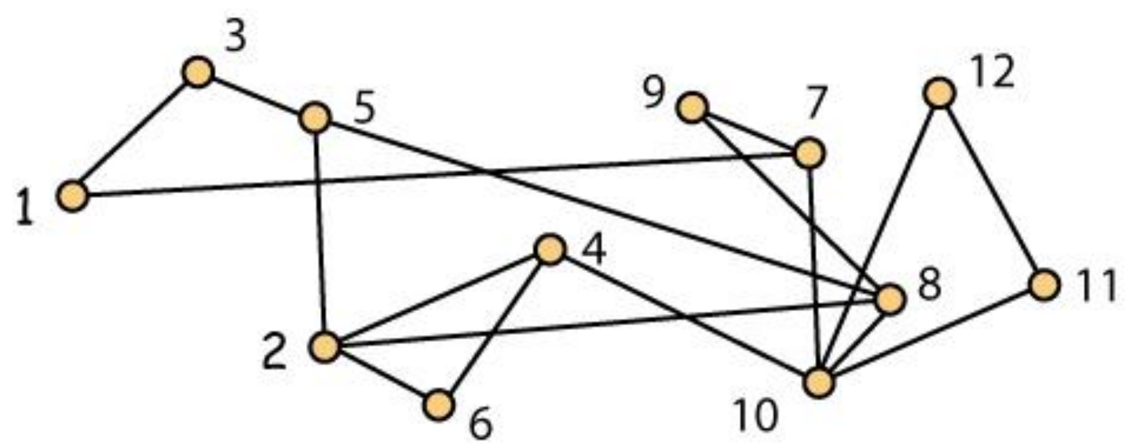
1. Vertices are also called nodes, points, locations, stations, etc.
2. Edges are also called arcs, lines, links, pipes, connectors, etc.
3. Remember that mathematicians are *selectively* lazy so when there is no confusion, an edge {x, y} will be denoted as xy. This can create some confusion when vertices are positive integers as how would one interpret a comment such as "consider the edge 2786".

Notation and Terminology (2)

1. When xy is an edge in G, we say x and y are adjacent in G. Alternatively, we say they are neighbors in G.
2. In a graph G, the set of all neighbors of a vertex x is denoted $N_G(x)$. And when the graph G is fixed in the discussion, this is typically abbreviated to just $N(x)$.
3. The integer $|N_G(x)|$ is called the degree of x in G, and is denoted $\deg_G(x)$. Again, when the graph is fixed, this is shortened to $\deg(x)$.

Notation and Terminology (3)

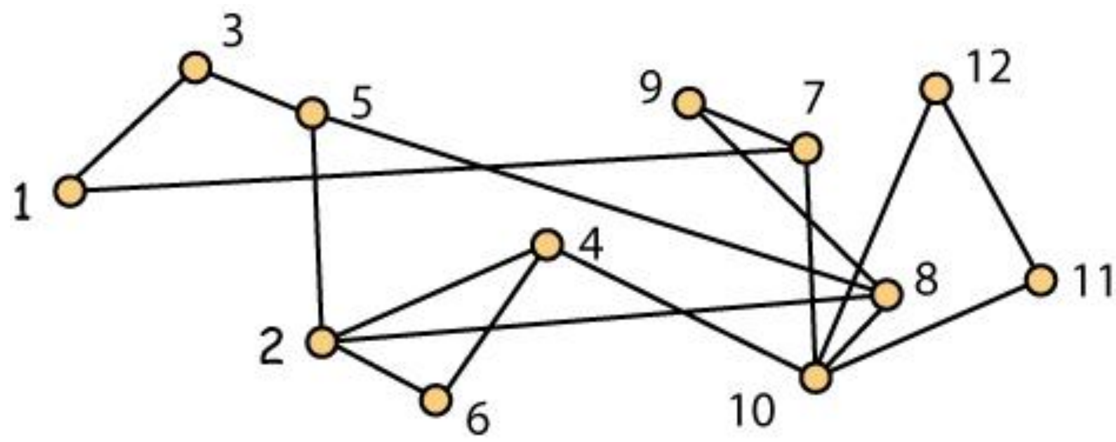
Example A graph with vertex set {1, 2, 3, …, 12}.



Questions Are 8 and 11 neighbors? What is deg(8)?

First Theorem in Graph Theory

Example Let G = (V, E) be a graph and let q be the number of edges in G. Then $\sum_{x \in V} deg_G(x) = 2q$



Exercise Verify this theorem for the graph illustrated above.

Carlos and Dave

Overheard in Conversation Dave said that he was working with a graph and carefully counted all the degrees and said here is full listing of all the values:

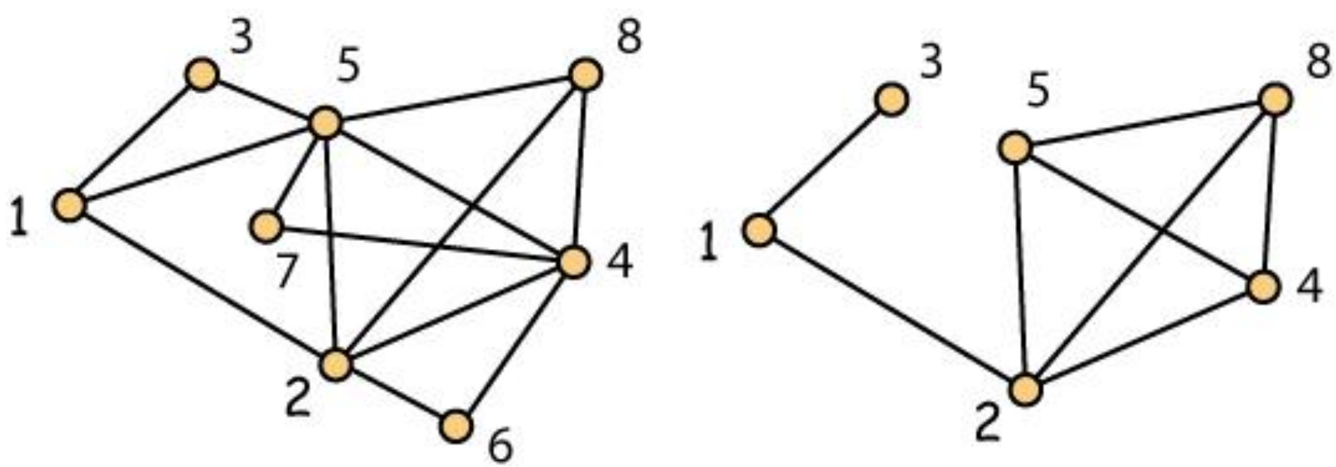
16, 13, 12, 18, 16, 22, 11, 16, 14, 10, 8, 12, 14, 16, 8, 7, 10, 20, 12, 14, 16, 8, 6, 6, 8, 4, 8, 6, 6, 6, 6, 8, 10, 5, 8, 8, 6, 6, 6, 6, 3, 6, 4, 8, 8, 8, 4, 8, 10, 12

Carlos remarked gently "Perhaps you should check your work."

The Notion of a Subgraph

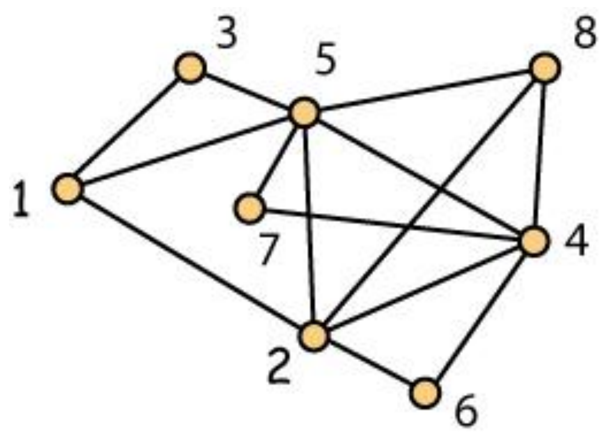
Definition A graph $G' = (V', E')$ is a subgraph of a graph $G = (V, E)$ when V' is contained in V and E' is contained in E.

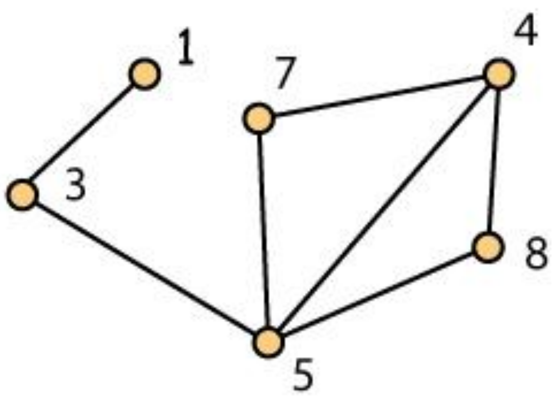
Example On the left, we show a graph with vertex set $\{1, 2, \ldots, 8\}$. The graph on the right is a subgraph.



The Notion of a Subgraph

Question We show a graph G with vertex set {1, 2, …, 8} on the left. Is the graph on the right a subgraph?

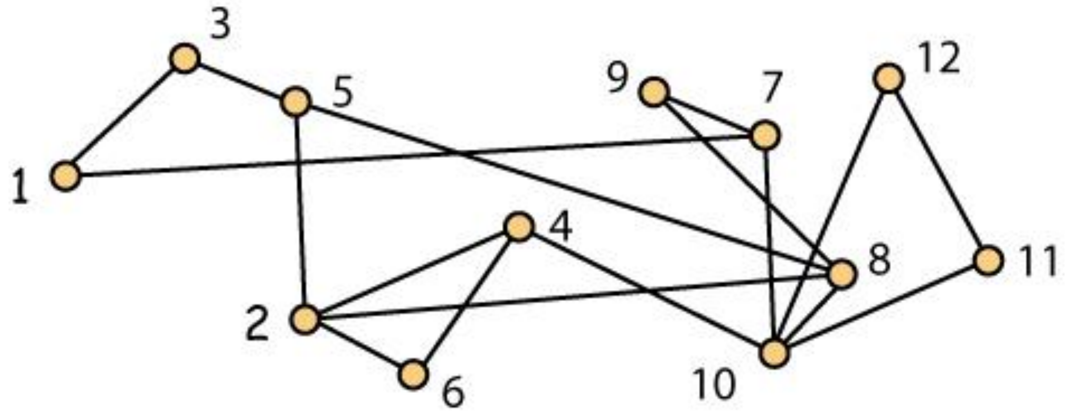




Paths in Graphs

Definition Let $G = (V, E)$ be a graph. When $n \geq 1$, a sequence $P = (x_1, x_2, \ldots, x_n)$ of n distinct vertices in G is called a path from x_1 to x_n in G if x_i is adjacent to x_{i+1} in G whenever $1 \leq i < n$.

Example In the graph shown, (7, 9, 8, 5, 2, 6, 4) is a path from 7 to 4.



Size of Paths

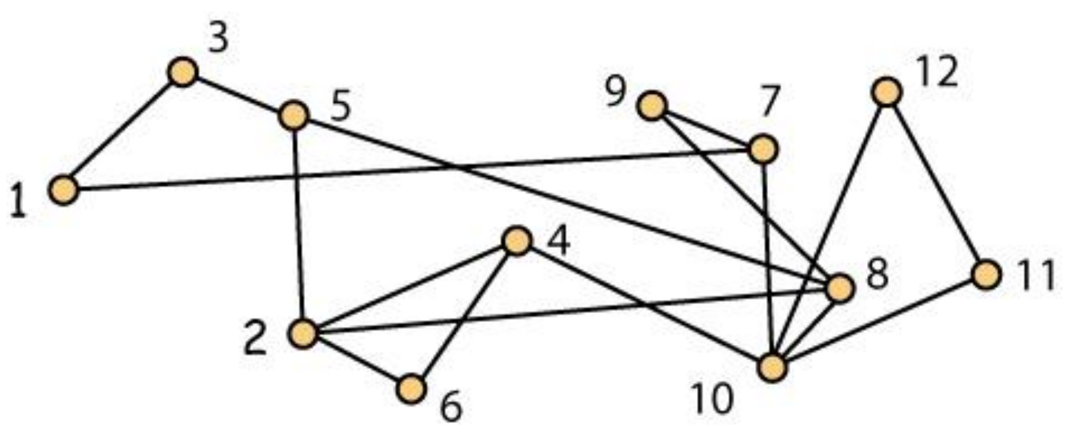
Convention Many authors measure how big a path is in terms of the number of edges, so they will say that a path (a, b, c, d, e) from a to e has length 4. In particular, they would say that when x and y are neighbors, the path (x, y) has length 1. Other authors prefer to measure paths in terms of the number of vertices, so they would say that the path (a, b, c, d, e) has size 5. We prefer the second option, so we will always talk about paths of a certain size and this will count the number of vertices and not the number of edges.

But in about two months, we will change our minds?!!

Connected Graphs

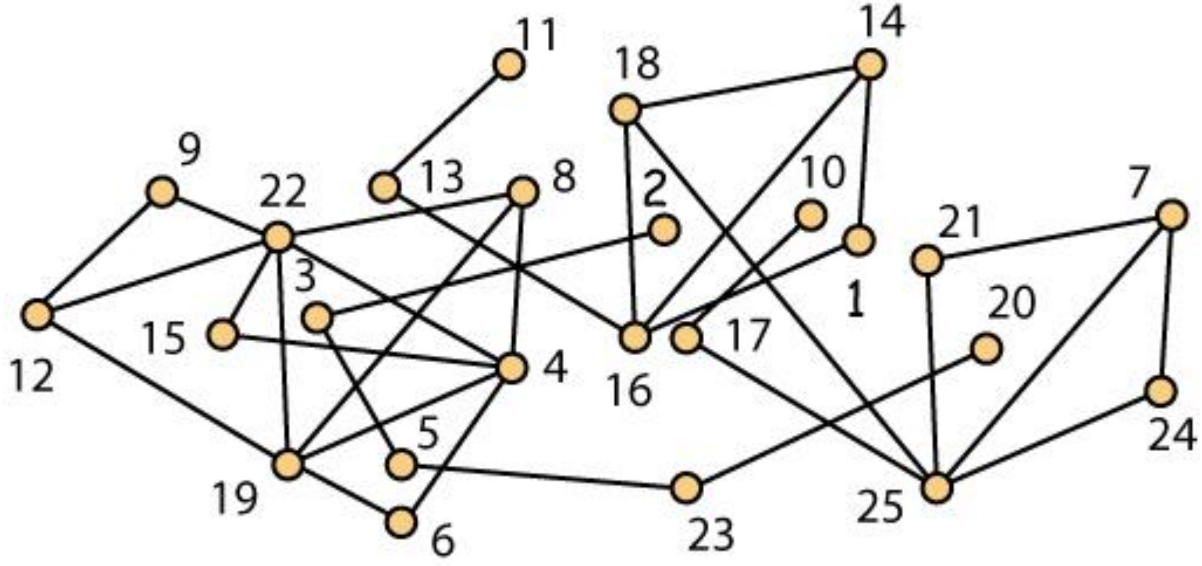
Definition Let $G = (V, E)$ be a graph. We say G is connected if for all x, y in V with $x \neq y$, there is a path from x to y in G.

Example The graph shown below is connected.



Connected Graphs (2)

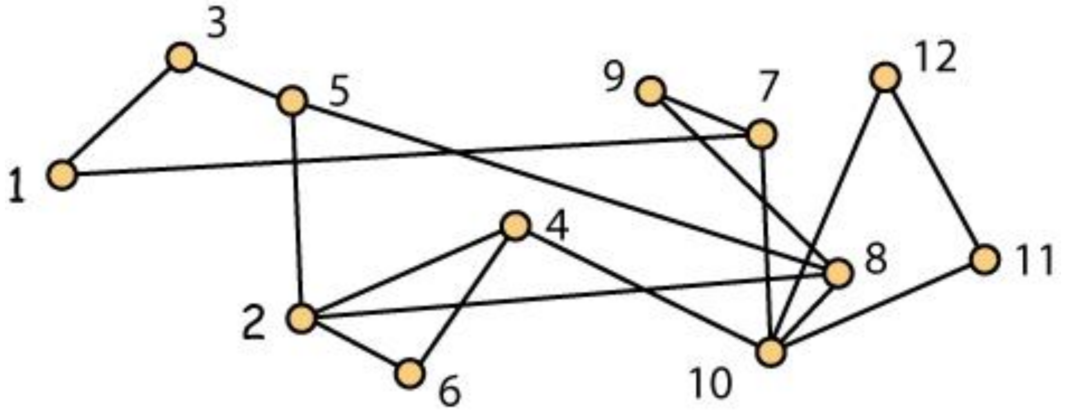
Definition Let $G = (V, E)$ be a disconnected graph. A subgraph $H = (V', E')$ of G is called a component of G if H is connected and any subgraph of G which contains H properly is disconnected. Is this graph connected? If not, how many components does it have?



Cycles in Graphs

Definition Let $G = (V, E)$ be a graph. When $n \geq 3$, a sequence $P = (x_1, x_2, \ldots, x_n)$ of n distinct vertices in G is called a cycle of length n in G if x_i is adjacent to x_{i+1} in G whenever $1 \leq i < n$ and x_n is adjacent to x_1 in G.

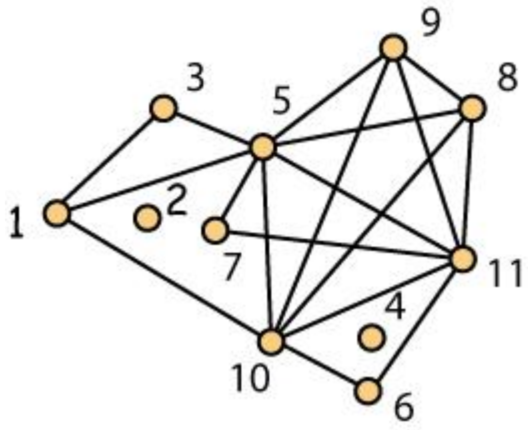
Example In the graph shown, (5, 8, 9, 7, 1, 3) is a cycle of length 6.



Loose Points in Graphs

Definition A vertex x in a graph G is called a loose point (also an isolated point) if it has no neighbors, i.e., $\deg_G(x) = 0$.

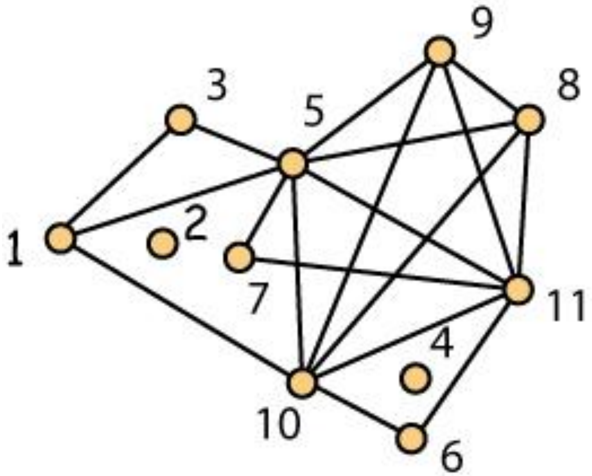
Example Below we show a graph with vertex set {1, 2, …, 11}. In this graph, vertices 2 and 4 are loose points.



Cliques in Graphs

Definition Let $G = (V, E)$ be a graph. When $n \geq 1$, a set S of vertices in G is called a clique if any two distinct vertices in S are adjacent in G.

Example In this graph, the subsets {2}, {6, 10}, {1, 3, 5} and {5, 8, 9, 10, 11} are cliques. There are many more.



Xing and Zori

Overheard in Conversations Xing is a very good programmer and remarked to Zori that he could easily detect whether a large graph was connected and if it was disconnected whether it had any loose vertices. Zori was not impressed as she couldn't see any reason why anybody would care about either issue. Still, moderately annoyed with Xing's enthusiasm, she asked him about a problem she had read about on the web: Could he tell whether a graph on $2n$ vertices had a clique of size n. Xing hadn't thought about it … but now that he was challenged, he said he thought he could.

Questions for Thought

Challenges or Not? Given a graph $G = (V, E)$ with $|V| = n$, which of the following problems is easy and which is hard?

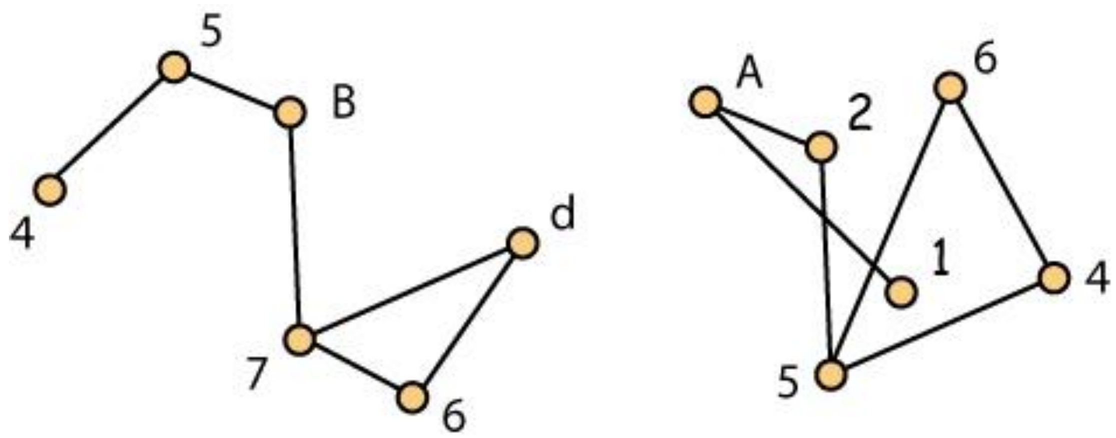
1. Is G connected?
2. Does G have a path on at least $n/2$ vertices?
3. Does G have a cycle of size at least $n/2$?
4. Does G have a clique of size at least $n/2$?

Also Suppose Alice and Bob are arguing about the correct answers to these questions for a graph with $n = 10{,}000$. Would you rather defend a "yes" answer or a "no" answer.

Isomorphic Graphs

Definition Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic when there is a bijection $f: V_1 \to V_2$ so that $\{x, y\}$ is an edge in G_1 if and only if $\{f(x), f(y)\}$ is an edge in G_2.

Exercise Show that the two graphs shown below are isomorphic.



Another Question for Thought

Challenge or Not? Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is it hard to tell whether they are isomorphic? If Yolanda says "yes" and Bob says "no", who has the easier task to convince an impartial referee?