October 6, 2015

Math 3012 – Applied Combinatorics Lecture 14

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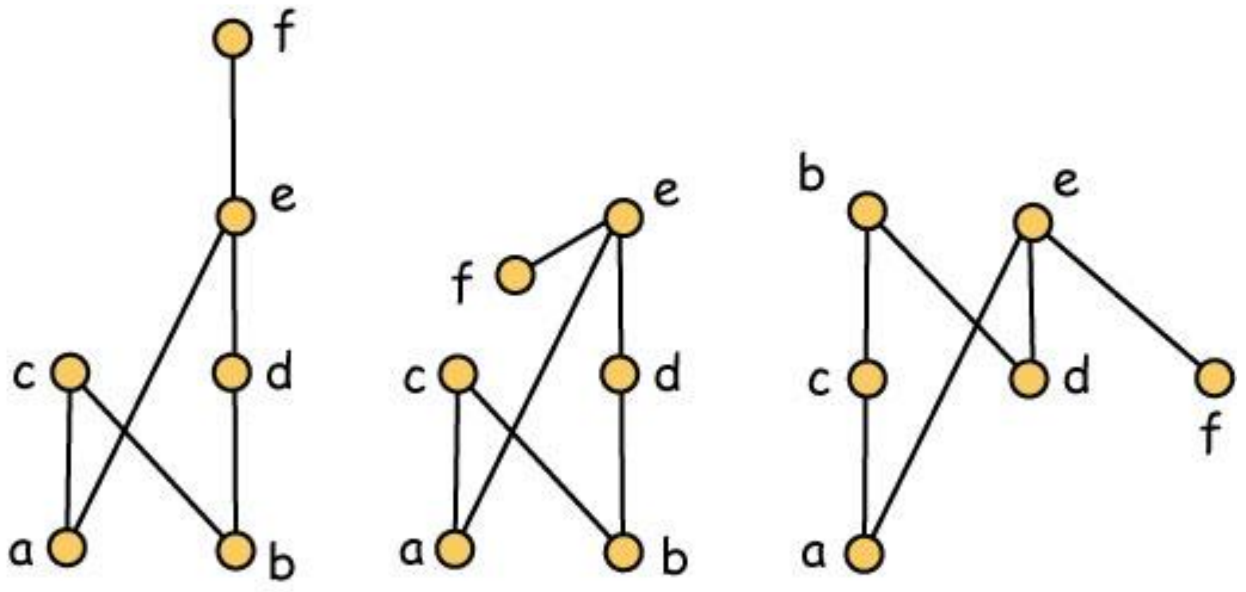
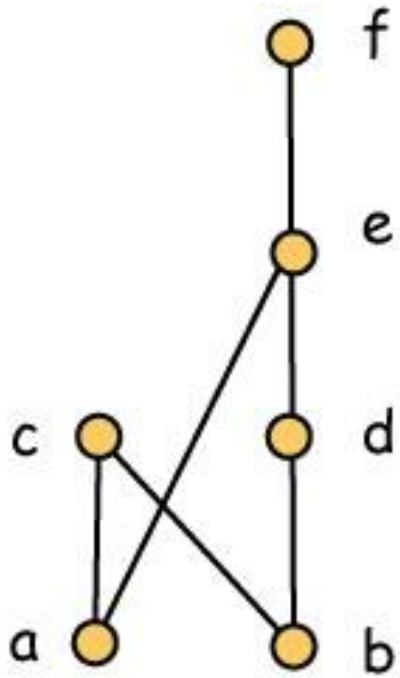
Three Posets with the Same Cover Graph

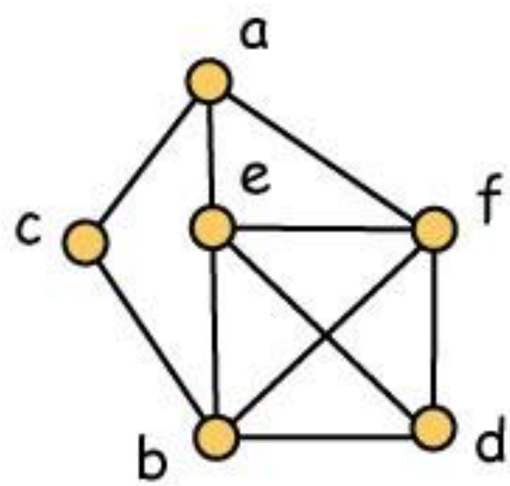


Exercise How many posets altogether have the same cover graph as these three?

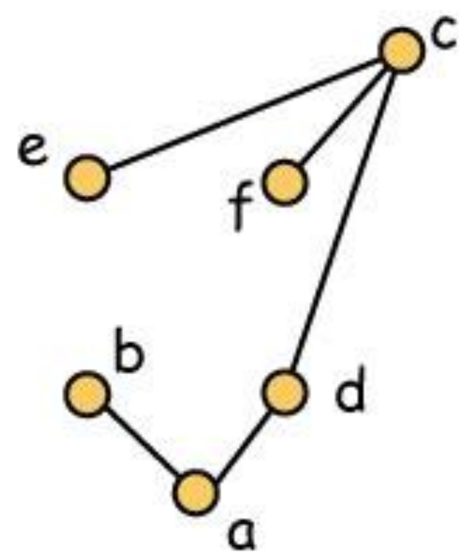
Comparability and Incomparability Graphs



Poset



Comparability Graph

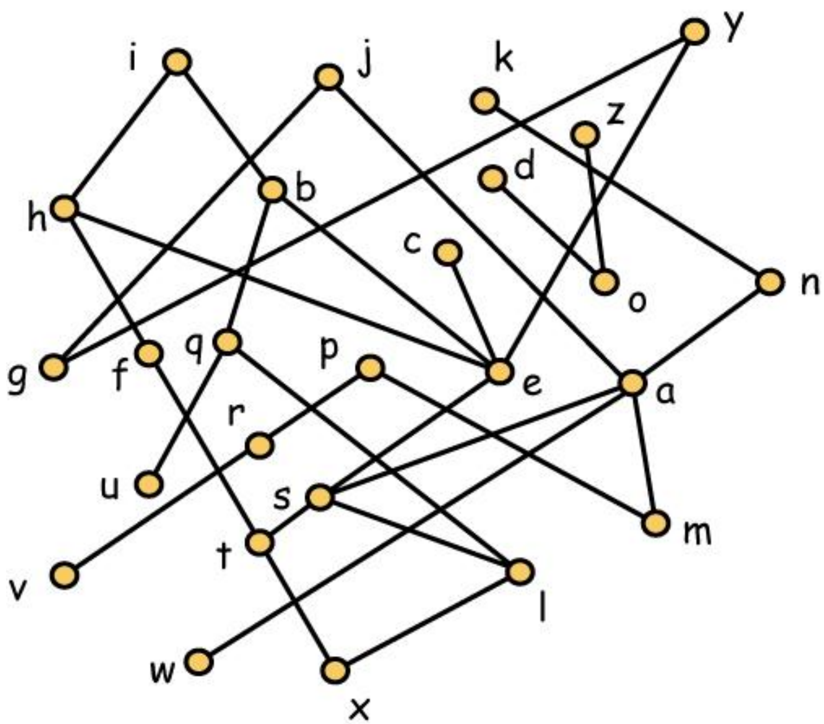


Incomparability Graph

Diagram for a Poset on 26 points

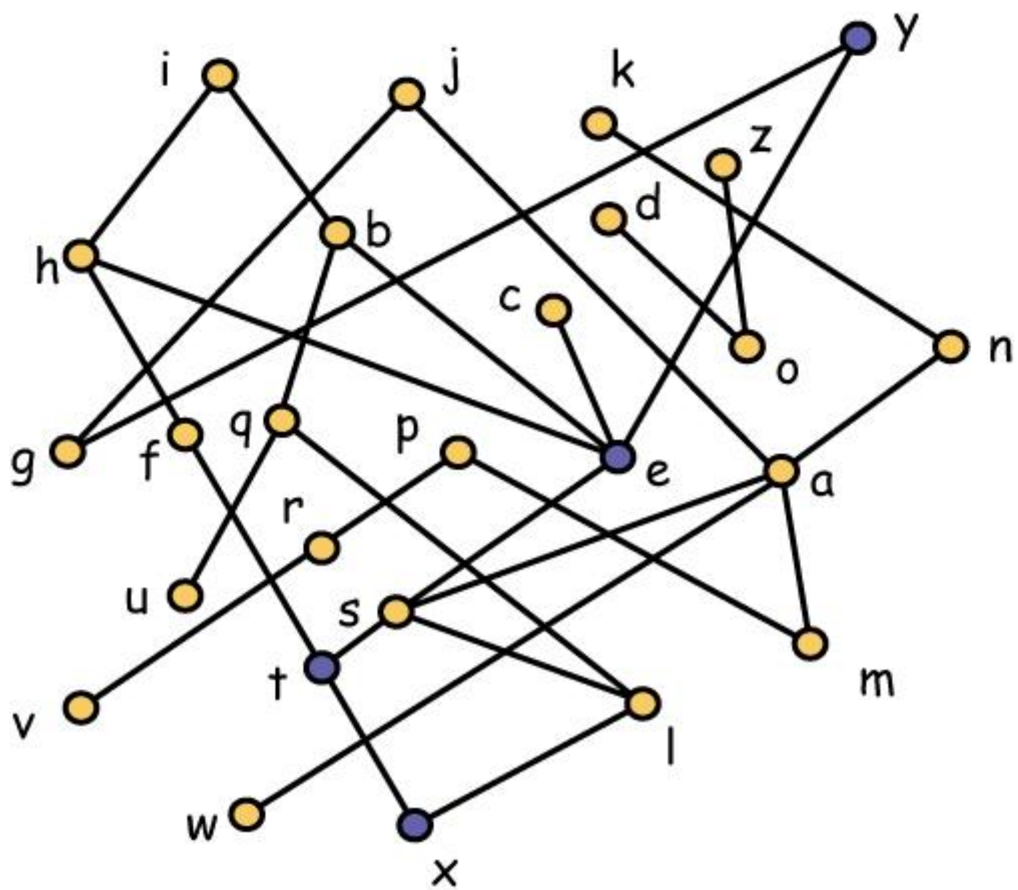
Terminology:

- b < i and s < y.
- j covers a.
- b > e and k > w.
- s and y are comparable.
- j and p are incomparable.
- c is a maximal element.
- u is a minimal element.



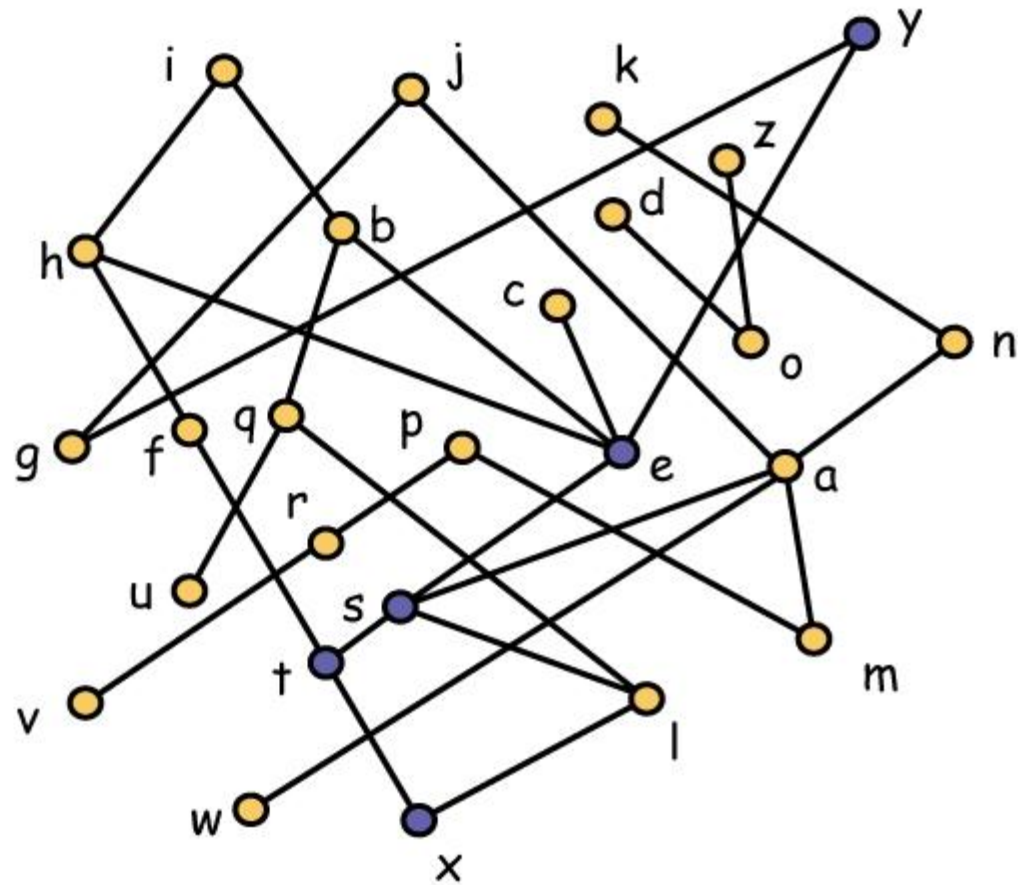
A Chain of Size 4

Definition *A* chain is a subset in which every pair is comparable.



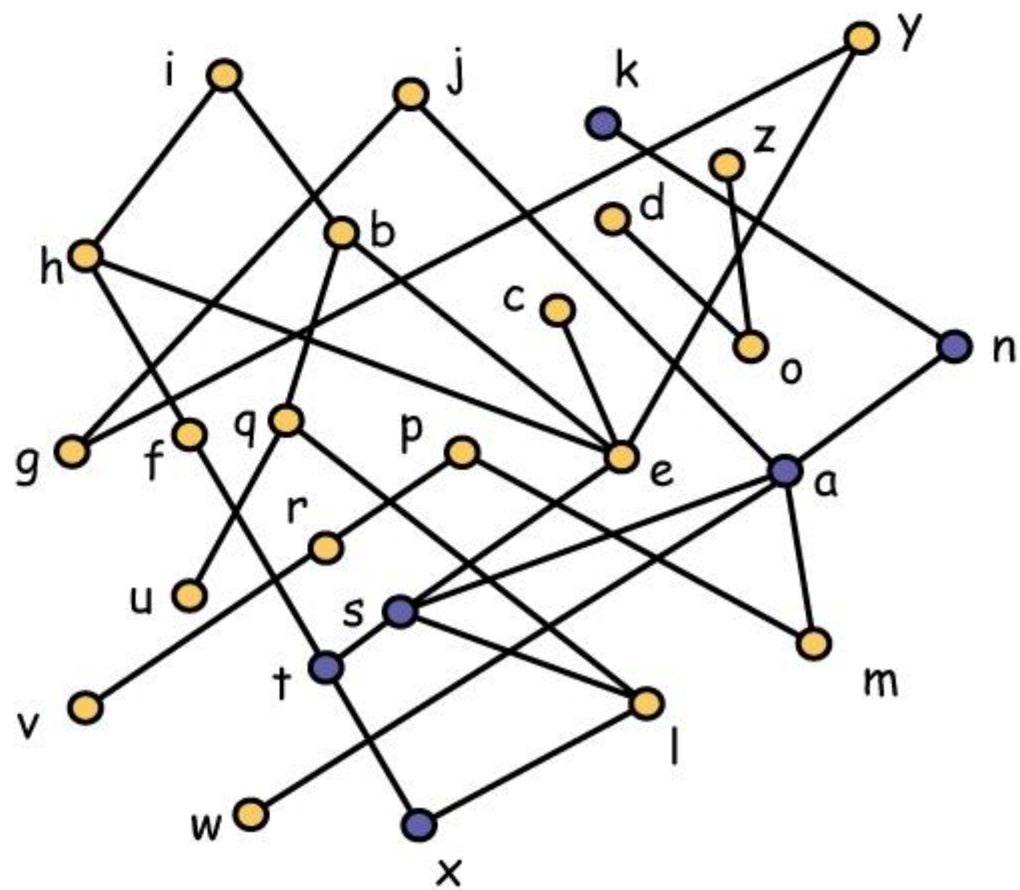
A Maximal Chain of Size 5

Definition *A* chain is maximal when no superset is also a chain.



A Maximal Chain of Size 6

Definition A chain is maximal when no superset is also a chain.



Height of a Poset

Definition The height of a poset P is the maximum size of a chain in P.

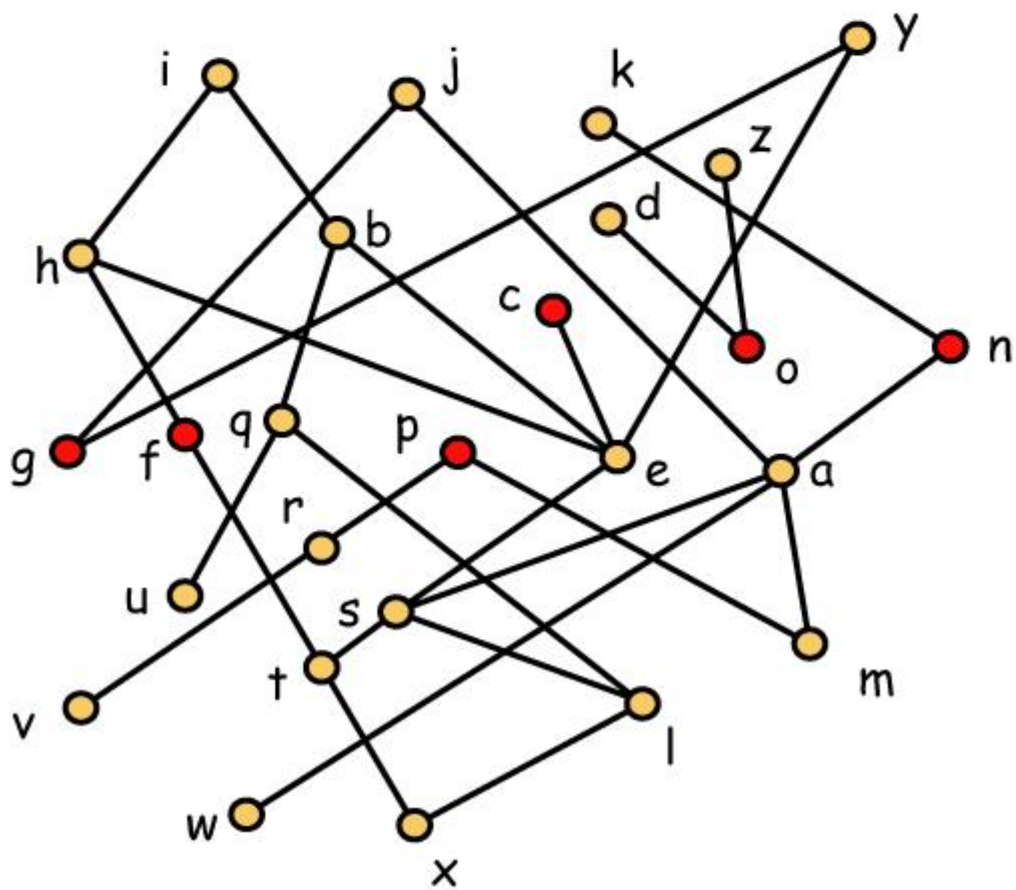
Question How hard is it to find the height of a poset?

Observation It is clear how to provide a certificate for the assertion: The height of P is at least h.

Question How would you provide a certificate for the assertion: The height of P is at most h.

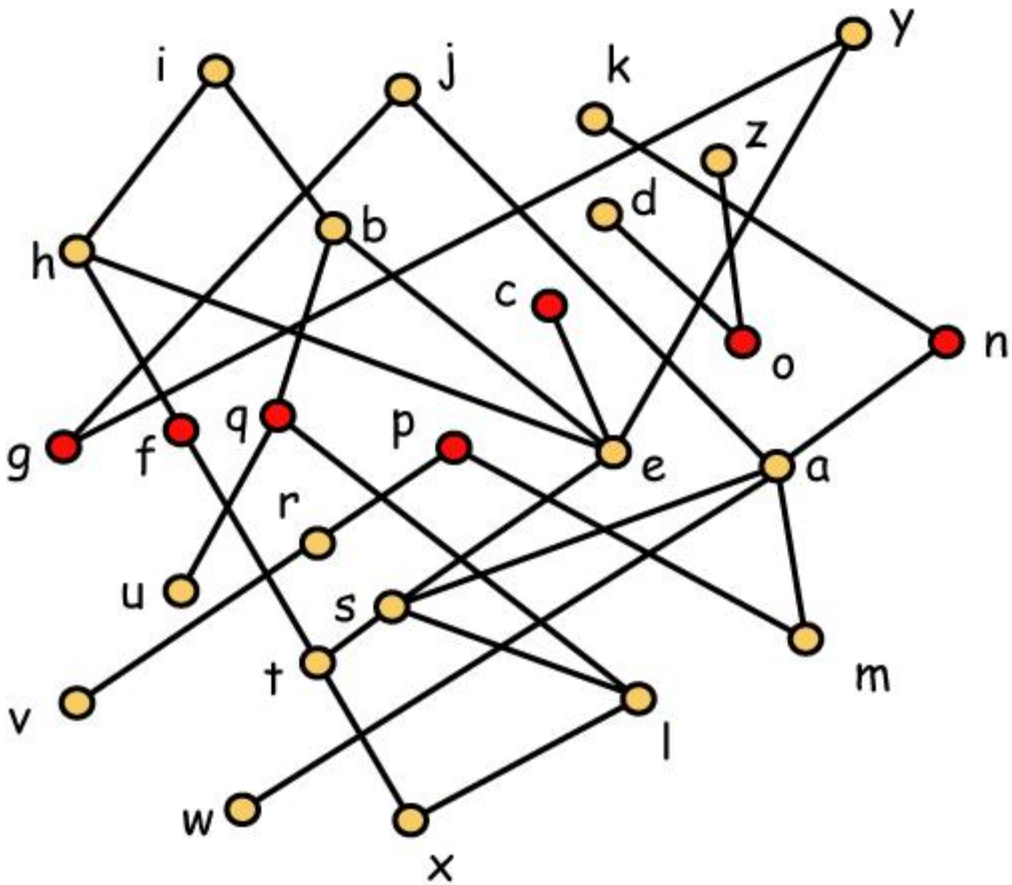
An Antichain of Size 6

Definition A subset is an antichain when every pair is incomparable.



A Maximal Antichain of Size 7

Definition An antichain is maximal when no superset is an antichain.



Width of a Poset

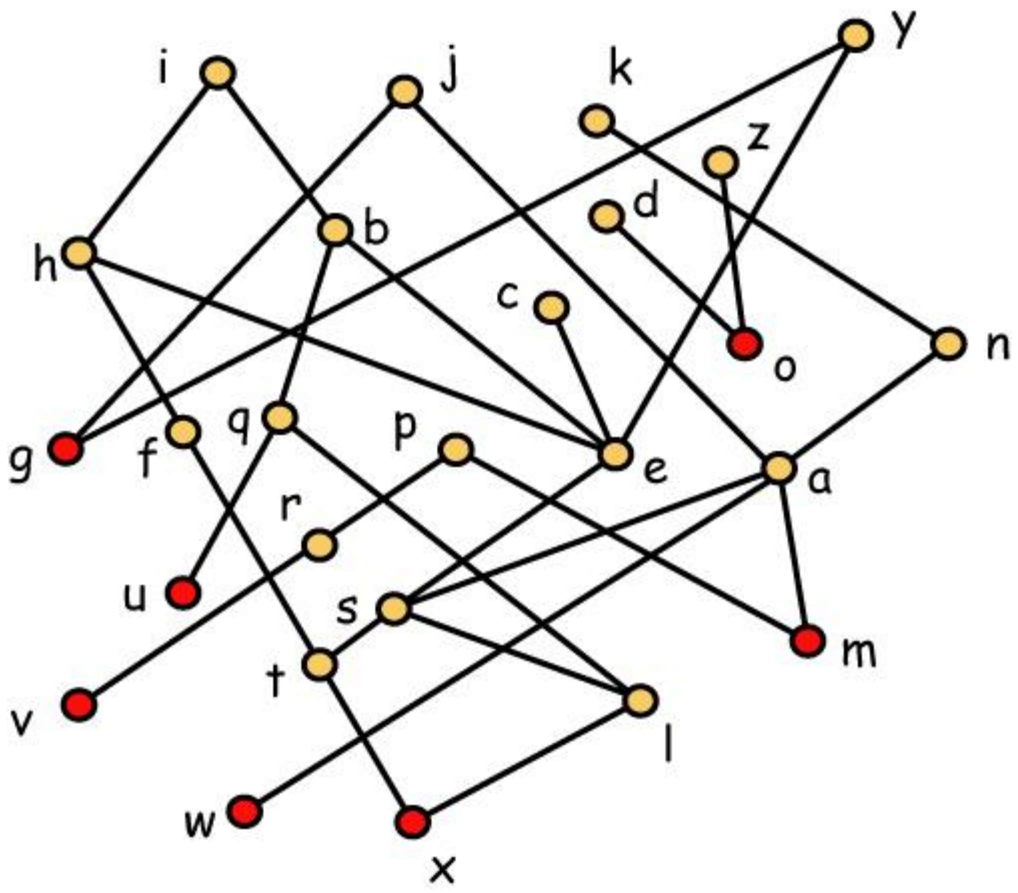
Definition The width of a poset P is the maximum size of an antichain in P.

Question How hard is it to find the width of a poset?

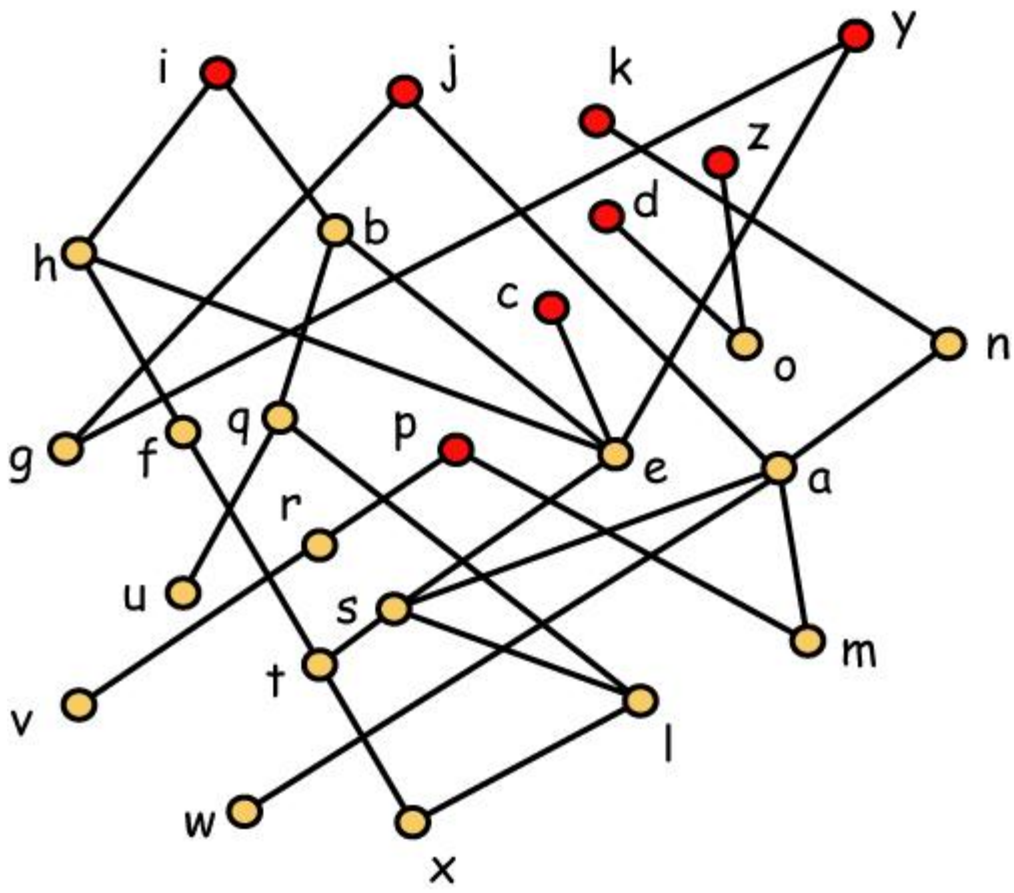
Observation It is clear how to provide a certificate for the assertion: The width of P is at least w.

Question How would you provide a certificate for the assertion: The width of P is at most w.

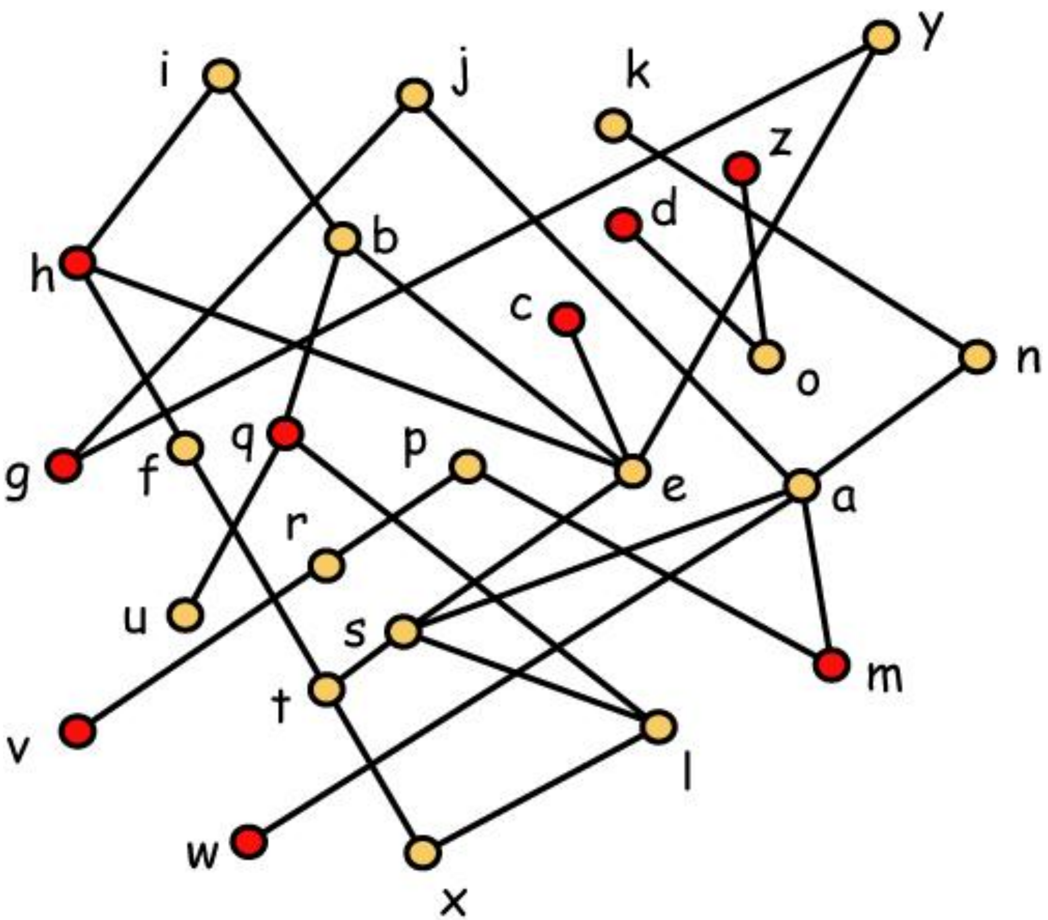
There are 7 Minimal Elements



There are 8 Maximal Elements



Width ≥ 9



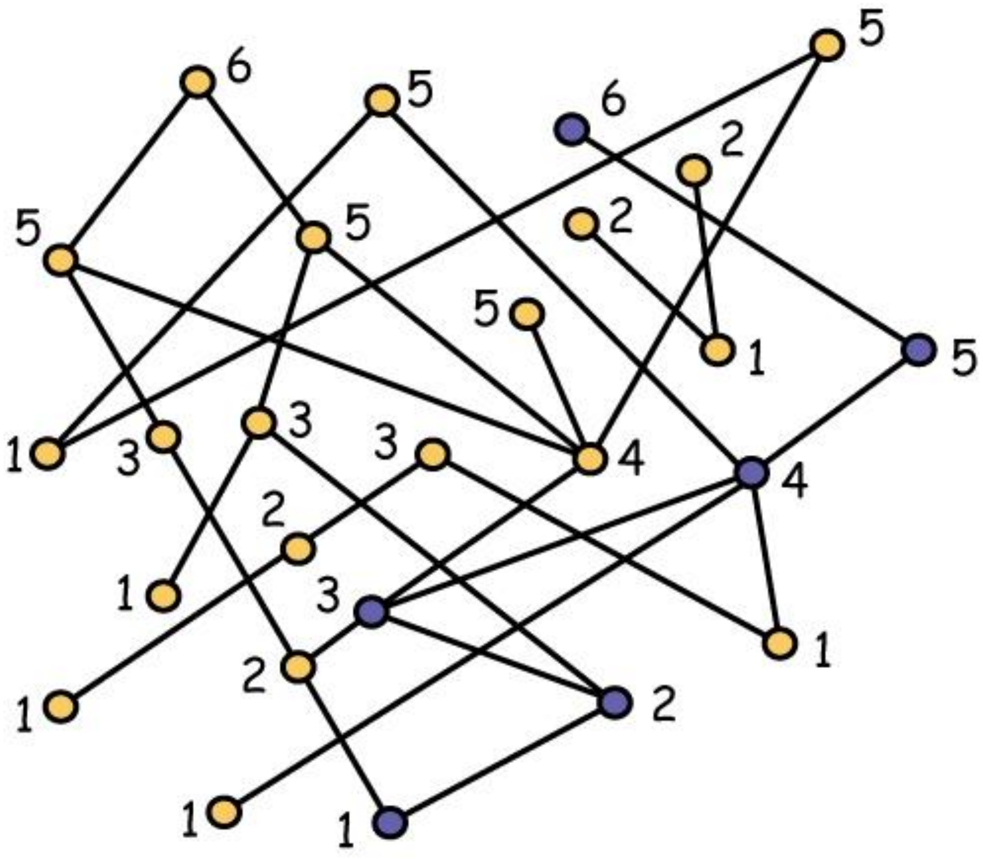
Certificates

Observation If P can be partitioned into t antichains, then the height of P is at most t.

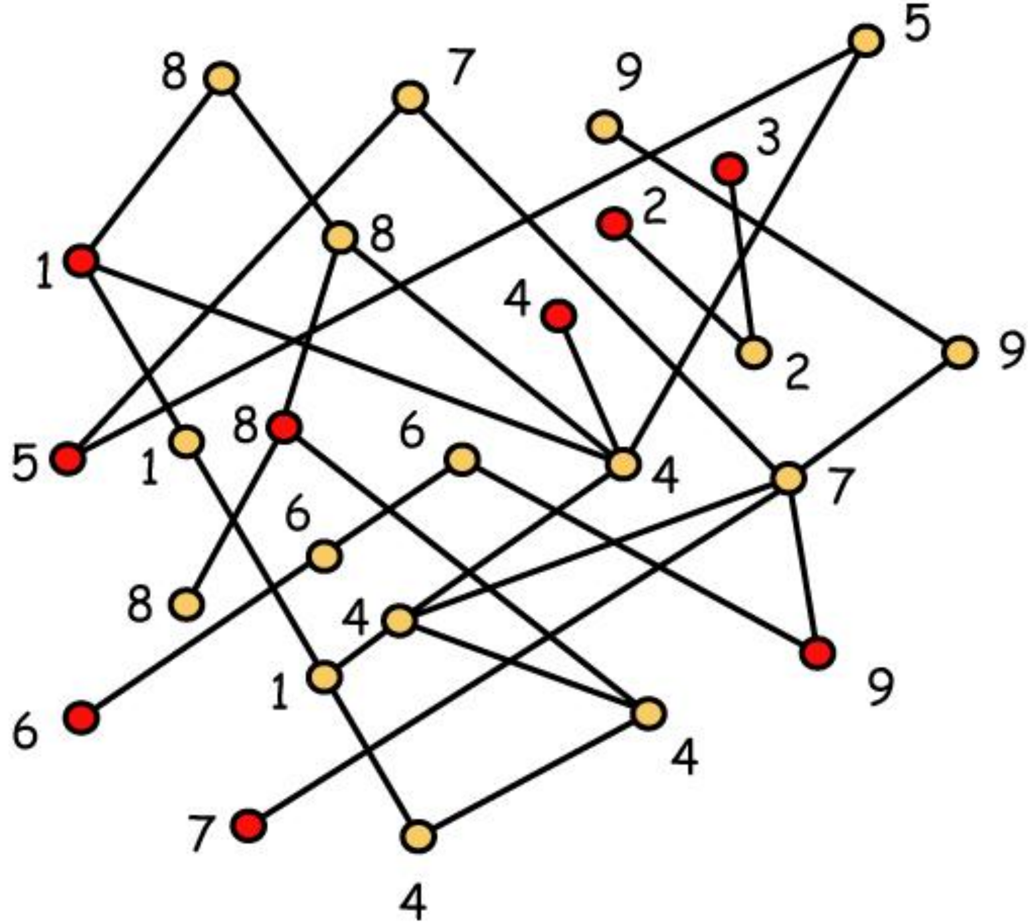
Observation If P can be partitioned into s chains, then the width of P is at most s.

Observation So that's how you would provide a certificate for the assertions: $\text{height}(P) \leq t$ or $\text{width}(P) \leq s$.

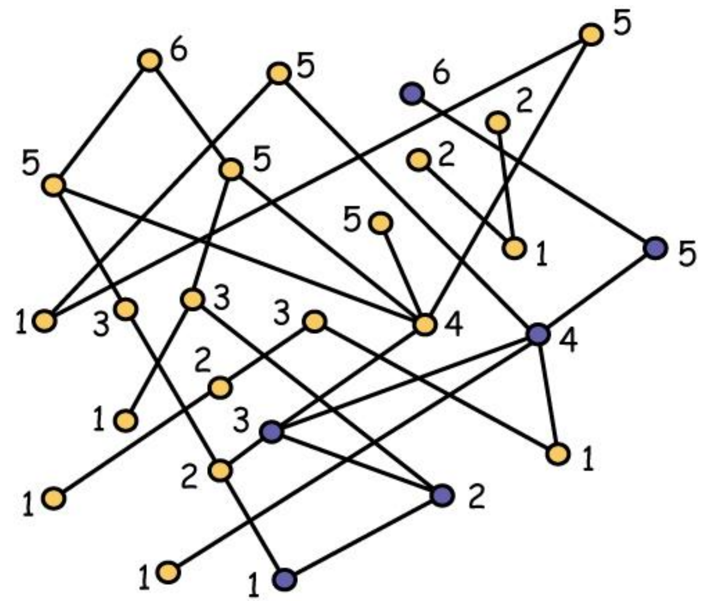
Height ≤ 6



Width ≤ 9



Mirsky's Theorem (Dual Dilworth)



Theorem (1971) A poset of height h can be partitioned into h antichains.

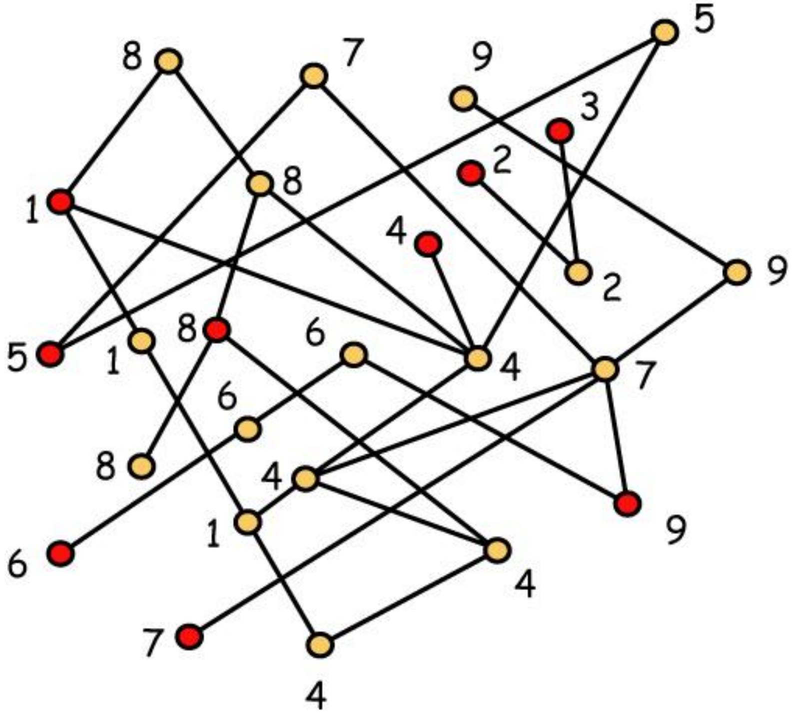
Proof Recursively remove the set of minimal elements.

Proof of Dual Dilworth

Proof For each i, let, A_i consist of those elements x from P for which the longest chain in P with x as its largest element has i elements. Evidently, each A_i is an antichain. Furthermore, the number of non-empty antichains in the resulting partition is just h, the height of P. Also, a chain C of size h can be easily found using back-tracking, starting from any element of A_h.

Algorithm A_1 is just the set of minimal elements of P. Thereafter, A_{i+1} is just the set of minimal elements of the poset resulting from the removal of $A_1, A_2, \ldots, A_i$.

Dilworth's Theorem



Theorem (1950) A poset of width w can be partitioned into w chains.

Proofs of Dilworth's Theorem

Fulkerson (1954) Used bipartite matching algorithm (network flows) to find minimum chain partition and maximum antichain simultaneously. We will study this right at the end of the course.

Gallai/Milgram (1960) Path decompositions in oriented graphs.

Perles (1963) Simple induction depending on whether there is a maximum antichain A with $U(A)$ and $D(A)$ non-empty. This is the proof found in most combinatorics textbooks.

The Proof of Dilworth's Theorem (1)

Proof True when width $w = 1$ and thus when $|P| = 1$. Assume valid when $|P| \leq k$. Then consider a poset P with $|P| = k + 1$.

For each maximal antichain A, let $D(A) = \{x : x < a$ for some a in $A\}$, and $U(A) = \{x : x > a$ for some a in $A\}$. Evidently, $P = A \cup D(A) \cup U(A)$ is a partition into pairwise disjoint sets.

The Proof of Dilworth's Theorem (2)

Case 1 There exists a maximum antichain A with both $D(A)$ and $U(A)$ non-empty.

Label the elements of A as $a_1, a_2, \ldots, a_w$. Then apply the inductive hypothesis to $A \cup D(A)$, which has at most k points, since $U(A)$ is non-empty. WLOG, we obtain a chain partition $C_1, C_2, \ldots, C_w$ of $A \cup D(A)$ with a_i the greatest element of C_i for each $i = 1, 2, \ldots, w$.

The Proof of Dilworth's Theorem (3)

Then apply the inductive hypothesis to $A \cup U(A)$. WLOG, we obtain a chain partition $C'_1, C'_2, \ldots, C'_w$ with a_i the least element of C'_i for each i. Then $C_i \cup C'_i$ is a chain for each $i = 1, 2, \ldots, w$ and these w chains cover P.

Case 2 For every maximum antichain A, at least one of $D(A)$ and $U(A)$ is empty.

Choose a maximal element y. Then choose a minimal element x with $x \leq y$ in P. Note that we allow $x = y$. Regardless, $C = \{x, y\}$ is a chain – of either one or two points - and the width of $P - C$ is $w - 1$. Partition $P - C$ into $w - 1$ chains, and then add chain C to obtain the desired chain partition of P.

Historical Notes

1. Gallai & Milgram published their work in 1960, but they had the result much earlier (in the late 1940's) before Dilworth's theorem was published.
2. But Dilworth knew the chain partitioning theorem much earlier too, so it remains accurate to attribute the result to Dilworth.
3. Dilworth, Fulkerson, Gallai & Milgram and many others also knew the dual form of Dilworth's theorem: a poset of height h can be partitioned into h antichains, also from the 1940's. But evidently, all of them considered the result too trivial to write down.
4. So today many people just refer to the dual result as "dual Dilworth" and don't make an attribution.