ECE 3077, Summer 2014

Homework #4

Due Thursday June 19, in class

Reading: B&T Chapter 2

1. Using you class notes, prepare a 1–2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Consider the following game. You give me \$1, and then choose a number between 1 and 6; call your choice C. I roll three dice. If none of the dice come up C, you get nothing back; if one of them comes up C you get \$2; if two come up C, you get \$3; if all three come up C you get \$4. What is your expected gain/loss?

3. Suppose that X is a Poisson random variable with parameter λ. As discussed in the notes, this means that the pmf for X is

$$p_X(k) = \mathrm{e}^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

   (a) Calculate $\mathrm{E}\left[\frac{1}{X+1}\right]$. You should be able to simplify things to get a nice closed form expression, much like what we did for $\mathrm{E}[X]$ on page II.14 of the notes.

   (b) It is a fact that the binomial pmf can be very closely approximated by a Poisson pmf when n is large and p is small, as

$$\mathrm{e}^{-\lambda}\frac{\lambda^k}{k!} \approx \frac{n!}{k!(n-k)!}p^k(1-p)^{n-k},$$

   with $\lambda = np$, provided that $k \ll n$, n is very large, and p is very small. What is the value of the two expressions above when $n = 100$, $p = .01$ and $k = 0, 1, 2, 3, 4, 5$? Plot your answers (as a function of k) using the stem command in MATLAB.

4. The MegaMillions Lottery[1], for which you can buy a ticket for \$1 at any convenience store in Georgia (or in 43 other states), works as follows: five numbers between 1 and 56 are drawn without replacement (these are the "white balls" taken from the same pool), and then an additional number between 1 and 46 is drawn independently (the "mega ball" taken from its own pool). To win the jackpot, you must choose the first 5 numbers (in any order) and the last number correctly.

[1] See also `http://en.wikipedia.org/wiki/Mega_Millions`, which also gives the answer for part (a) if you get stuck.

(a) Suppose you purchase a ticket using the QuickPick option, which chooses a valid configuration of numbers uniformly at random from among all the possibilities. What the probability p that you hold a winning ticket? (Keep your answer in fractional, not decimal, form.)

(b) Suppose that n people buy tickets using QuickPick. What is the probability that k of them chose the winning number? (Hint: Binomial.)

(c) If more than one person picks the winning numbers, they split the jackpot equally among them. Suppose the jackpot is \$350 million, and 200 million tickets are sold (again, all using QuickPick). Conditioned on the event that you win, what is the expected payout?
(You are tempted to compute the average number of people that picked the same number as you, and then divide \$350 million by this number. But you know that is wrong since $\mathrm{E}[1/(X+1)] \neq 1/(\mathrm{E}[X]+1)$. Be smart. Use the Poisson approximation and expectation calculation from question 3 in this homework.)

(d) With the jackpot at \$350 million and 200 million tickets sold, what is your expected gain when you buy a ticket? (If you guess the wrong number, you lose a dollar. If you win, you gain some large amount of money that depends on how many people you have to split it with.)

5. Each 20 oz bottle of Sprite comes with a one in six chance of winning another bottle of Sprite, which itself comes with a one in six chance of winning yet another bottle, etc. What is the expected number of ounces of Sprite received for a single purchase?

6. M&Ms come in six different colors; within a bag of any size, the color of each M&M is uniformly distributed between these six colors, and is independent of the colors of the other M&Ms.

   You are drawing M&Ms one at a time from a very large bag and observing their color.

   (a) Suppose that all of the M&Ms you have drawn out so far only have two distinct colors. Let X be the discrete random variable defined as

   $$X = \text{number of additional draws required to obtain a 3rd distinct color.}$$

   Determine the pmf and mean of X.

   (b) Suppose that all of the M&Ms you have drawn out so far have n distinct colors, where $1 \leq n \leq 5$. Define

   $$X = \text{number of additional draws required to obtain the } (n+1)\text{th color.}$$

   Determine the pmf and mean of X.

   (c) Now let

   $$X = \text{total number of draws required to obtain all 6 colors.}$$

   Calculate $\mathrm{E}[X]$.
   (Hint: Let X_1 be the number of draws to get one color, X_2 be the number of draws to get the second color after the first is in hand, X_3 be the number of draws to get the third color after the first two are in hand, etc. Realize that $\mathrm{E}[X] = \mathrm{E}[X_1 + X_2 + \cdots + X_6] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_6]$.)

7. Suppose you are playing the following very favorable game of "heads and tails". A fair coin is flipped until it comes up tails. If it comes up tails on the qth flip, you receive \$$2^q$.

   (a) What is the expected number of flips before tails?

   (b) What is your expected payout?

   (c) Write a MATLAB program that simulates 10,000 trials of this game, and record the payout in vector `P` (so `P` will have 10,000 entries). Turn in your code, and a (clearly labeled) stem plot of `P`.

   (d) Survey question (no right or wrong answer): How much would you pay for the privilege of playing this game one time? Ten times? One hundred times? Think about it, then express your feelings.

8. **A Monte Carlo Estimate of π.** In this problem, we will explore how one might build up an estimate of π just from a stream of independent random variables drawn on the unit square.

   Let (x_i, y_i) for $i = 1, 2, \ldots$ be a series of points chosen independently and according to the uniform law from the unit square $[0,1] \times [0,1]$. Define the Bernoulli random variable Z_i as

$$Z_i = \begin{cases} 1 & x_i^2 + y_i^2 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) What is the probability that $Z_i = 1$?

   (b) Given N independent realizations $Z_1, Z_2, \ldots, Z_N$, let $\bar{Z}_N$ be the random variable

$$\bar{Z}_N = \frac{1}{N}\left(Z_1 + Z_2 + \cdots + Z_N\right).$$

   What is the expected value $\mathrm{E}[\bar{Z}_N]$? How can we choose $\alpha \in \mathbb{R}$ such that setting $W_N = \alpha \bar{Z}_N$ results in $\mathrm{E}[W_N] = \pi$?

   (c) Calculate $\mathrm{var}(W_N)$. How large must N be so that $\mathrm{var}(W_N) \leq 10^{-4}$?

   (d) Using MATLAB, create $1,000,000$ independent realizations of W_N with $N = 100$ and $N = 1000$; store then in a vector `WN`. Use the MATLAB command `hist` to get a histogram of your realizations; I will suggest `hist(WN,20)`. Plot out histograms for both values of N. On your plots, label the mean μ, and $\mu \pm \sigma$, $\mu \pm 2\sigma$, $\mu \pm 3\sigma$, where σ is the standard deviation of W_N.

   Note that N independent random variables drawn from the interval $[0,1]$ according to the uniform law can be generated in MATLAB using `rand(N,1)`.

   As always, turn in your code.

   (e) Describe, in qualitative terms, how accurate an estimate W_N is of π.