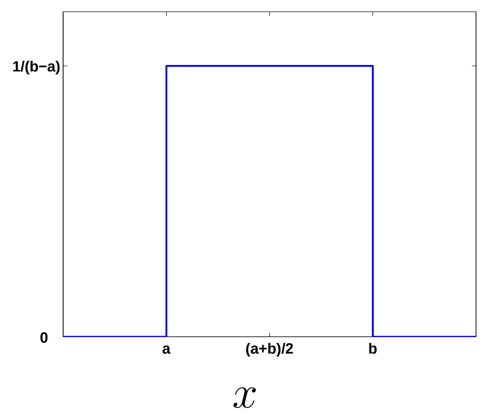
Compendium of continuous random variables

In this set of notes, we collect the basic facts (pdf, cdf, mean, variance, etc) for random variables that are commonly used in probabilistic modeling and statistics. All of the distributions discussed below also have Wikipedia pages, where more interesting facts can be found.

Uniform distribution

We say that X is **uniform** on $[a, b]$ if

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

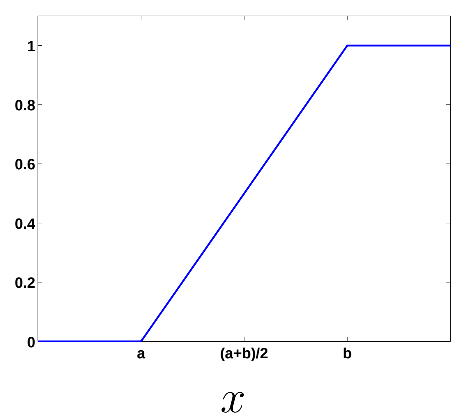


We write this as

$$X \sim \text{Uniform}([a, b]).$$

The cdf is

$$F_x(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b. \end{cases}$$



ECE 3077 Notes by M. Davenport, J. Romberg and C. Rozell. Last updated 9:26, June 17, 2014

The expectation and variance are

$$\mathrm{E}[X] = \frac{a+b}{2}, \qquad \mathrm{var}(X) = \frac{(b-a)^2}{12}.$$

These quantities are computed using basic calculus:

$$\mathrm{E}[X] = \frac{1}{b-a}\int_a^b x dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

and

$$\begin{aligned}\mathrm{E}[X^2] &= \frac{1}{b-a}\int_a^b x^2 dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b \\ &= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}\end{aligned}$$

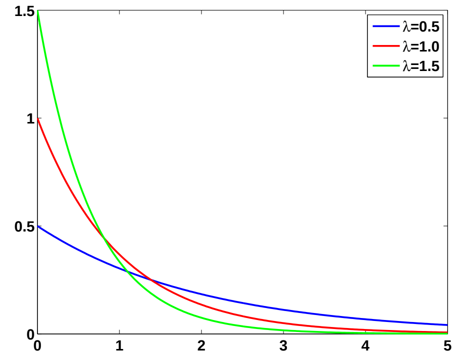
and so

$$\mathrm{var}(X) = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} = \frac{(b-a)^2}{12}.$$

Exponential distribution

We say that X is **exponential** if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$



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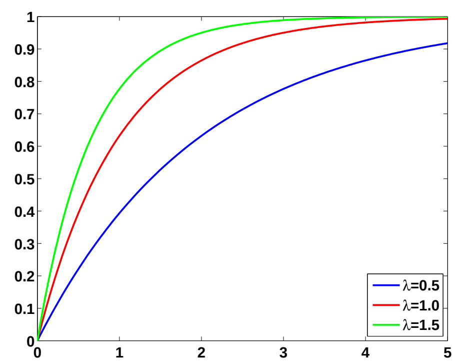
We write this as $X \sim \mathrm{Exp}(\lambda)$.

The exponential random variable is often used to model waiting times (i.e. the amount of time that will elapse before something occurs). The parameter λ is sometimes called the *rate*.

The cdf is

$$F_x(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$



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The mean and variance are given by

$$\mathrm{E}[X] = \frac{1}{\lambda}, \quad \mathrm{var}(X) = \frac{1}{\lambda^2}.$$

To calculate the mean, first write:

$$\mathrm{E}[X] = \int_0^\infty x\lambda e^{-\lambda x} dx.$$

Computing this quantity requires integration by parts:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx.$$

Take $u(x) = x$ and $v'(x) = \lambda e^{-\lambda x}$, which implies that $v(x) = -e^{-\lambda x}$. Then

$$\begin{aligned}
\mathrm{E}[X] &= \left[-xe^{-\lambda x}\right]_0^\infty - \int_0^\infty e^{-\lambda x} dx \\
&= 0 + \frac{1}{-\lambda}(0-1) \\
&= \frac{1}{\lambda}.
\end{aligned}$$

We can also calculate $\mathrm{E}[X^2]$ using integration by parts (with $u(x) = x^2$ and $v'(x) = -e^{-\lambda x}$:

$$\begin{aligned}
\mathrm{E}[X^2] &= \left[-x^2e^{-\lambda x}\right]_0^\infty - \int_0^\infty 2xe^{-\lambda x} dx \\
&= 0 + \frac{2}{\lambda}\,\mathrm{E}[X] \\
&= \frac{2}{\lambda^2}.
\end{aligned}$$

Thus

$$\mathrm{var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Exercise:
The time until the next train arrives is modeled by an exponential random variable with expected value 10 minutes (i.e., $\lambda = 1/10$). It is now 5pm. What is the probability that the train arrives between 5:01pm and 5:02pm?

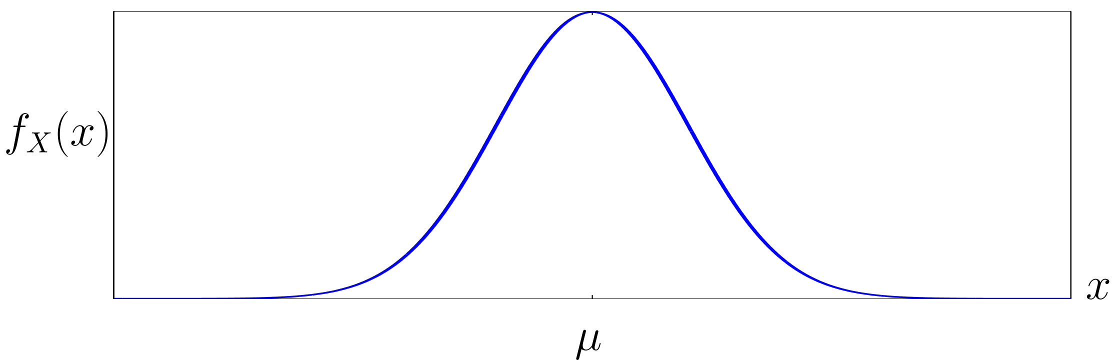
Normal distribution

We call a continuous random variable X **normal** or **Gaussian** if it has pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are parameters (it should be clear that we need $\sigma > 0$ for this to be a valid pdf).

The pdf is the classic "bell curve":



The location of the peak is μ, while the width is determined by σ.

In fact,

$$\mathrm{E}[X] = \mu, \qquad \mathrm{var}(X) = \sigma^2$$

It is easy to see that the pdf is symmetric around μ; this means that if the expectation exists (which it does), it must be μ.

To compute the variance, we need a little bit of calculus:

$$\mathrm{var}(X) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-\mu)^2\,\mathrm{e}^{-(x-\mu)^2/2\sigma^2}\;\mathrm{d}x$$

This integral looks hard, but we can make it manageable using the variable substitution:

$$y = \frac{x-\mu}{\sigma}, \qquad \text{so} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sigma},$$

then

$$\begin{aligned}
\mathrm{var}(X) &= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y^2\,\mathrm{e}^{-y^2/2}\;\mathrm{d}y \\
\text{(integrate by parts)} \qquad &= \frac{\sigma^2}{\sqrt{2\pi}}\left(-y\mathrm{e}^{-y^2/2}\right)\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2/2}\;\mathrm{d}y \\
&= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2/2}\;\mathrm{d}y \\
&= \sigma^2
\end{aligned}$$

where the last step follows from the fact that $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-y^2/2}$ corresponds to a pdf (a normal with mean 0 and variance 1) and hence must integrate to 1.

It is a fact that if X is normal with mean μ and variance σ^2, then

$$Y = aX + b, \qquad a, b \in \mathbb{R}$$

is also normal with

$$\mathrm{E}[Y] = a\mu + b, \qquad \mathrm{var}(Y) = a^2\sigma^2$$

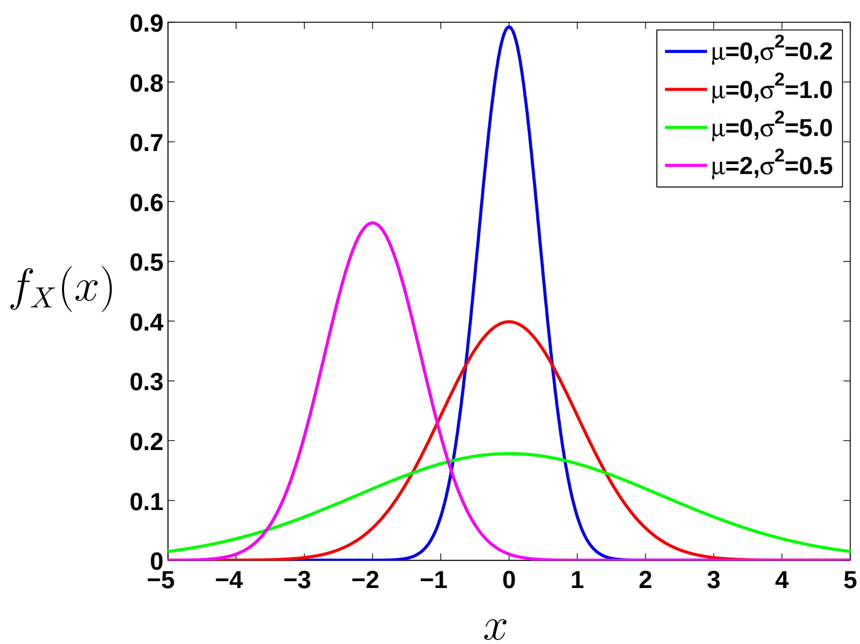
Notation: If X is normal with mean μ and variance σ^2, we write

$$X \sim \mathrm{Normal}(\mu,\ \sigma^2)$$

So with Y defined as above,

$$Y \sim \mathrm{Normal}(a\mu + b,\ a^2\sigma^2)$$

The pdf for different μ and σ^2 are plotted below.



Normal random variables are perhaps the most important class of random variables simply because they appear **everywhere**. Here is a nowhere-near-complete list of examples:

1. velocities of the molecules in an ideal gas

2. *Plinko*

3. position of a particle which undergoes diffusion

4. thermal noise in circuits

5. ambient noise in wireless communications

6. total score in an NBA game (approximate)[1]

7. scores in ESPN's annual *March Madness Tournament Challenge* (approximate)

8. height of American males ($\mu = 69.1$ inches, $\sigma = 2.9$ in) and American females ($\mu = 63.7$ in, $\sigma = 2.7$ in)

9. sum of a moderately large number of dice rolls (approximate)

10. number of people who will play the lottery tomorrow (approximate)

[1] The examples labeled "approximate" are really discrete random variables, but they can be very closely approximated by a continuous normal random variable.

The normal CDF

The cdf of a normal random variable X is

$$\mathrm{P}(X \le x) \;=\; \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x} \mathrm{e}^{-(u-\mu)^2/2\sigma^2}\,\mathrm{d}u$$

This is an important function, but unfortunately there is no closed-form expression for it.

To use it, we have look-up tables and MATLAB functions. For convenience, these tables/functions are in terms of a Normal(0, 1) random variables — this is called a **standard normal random variable**:

$$Y \sim \mathrm{Normal}(0,1).$$

We use $\Phi(y)$ to denote the cdf of Y:

$$\Phi(y) = \mathrm{P}(Y \le y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} \mathrm{e}^{-u^2/2}\,\mathrm{d}u.$$

Here is a plot:

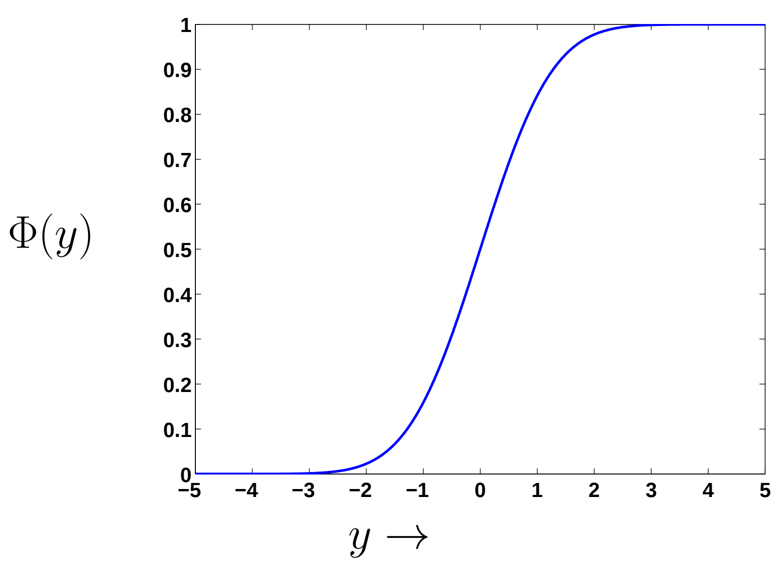


Table for $y \geq 0$:

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

For example,

$$
\begin{aligned}
\mathrm{P}(Y \leq 0) &= \Phi(0) = 0.5 \\
\mathrm{P}(Y \leq 1.42) &= \Phi(1.42) = 0.9222 \\
\mathrm{P}(Y \leq 0.68) &= \Phi(0.68) = 0.7517
\end{aligned}
$$

The table only contains entries for positive values of y, but we can get negative values by using a simple identity. Here is a particular example:

$$
\begin{aligned}
\Phi(-0.5) = \mathrm{P}(Y \leq -0.5) &= \mathrm{P}(Y \geq 0.5) \\
&= 1 - \mathrm{P}(Y < 0.5) \\
&= 1 - \Phi(0.5) \\
&= 1 - 0.6915 = 0.3085
\end{aligned}
$$

In general,

$$\Phi(-y) = 1 - \Phi(y)$$

We can use the lookup table for any normal random variable by using a simple transformation. It is an easily checked fact that if

$$X \ \sim \ \mathrm{Normal}(\mu, \sigma^2)$$

then

$$Y = \frac{X - \mu}{\sigma}$$

is also normal with

$$\begin{aligned} \mathrm{E}[Y] &= \mathrm{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\,\mathrm{E}[X] - \frac{\mu}{\sigma} = 0, \\ \mathrm{var}(Y) &= \frac{\mathrm{var}(X)}{\sigma^2} = 1. \end{aligned}$$

Then we can evaluate the cdf of X using

$$\begin{aligned} F_X(x) = \mathrm{P}\left(X \le x\right) &= \mathrm{P}\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right). \end{aligned}$$

Example. We can model the height of an American adult male as having a normal distribution with $\mu = 69.1$ inches and $\sigma = 2.9$ inches. What is the probability that an American male chosen at random will be 6' 2" or taller?

Answer: With $X \sim \mathrm{Normal}(69.1, (2.9)^2)$, we can compute the probability that the man is *shorter* than 6' 2" using

$$\mathrm{P}\left(X < 74\right) = \Phi\left(\frac{74 - 69.1}{2.9}\right) = \Phi(1.69) = 0.9545$$

and so

$$\mathrm{P}\left(\text{taller than 6' 2"}\right) = \mathrm{P}\left(X \ge 74\right) = 1 - \mathrm{P}\left(X < 74\right) = 0.0455,$$

or about a 4.5% chance.

Exercise:
The rainfall in February in Atlanta can be modeled as a normal random variable with mean 4.68 inches and standard deviation 3.2 inches. What is the probability that the total rainfall this February is less than 5 inches?

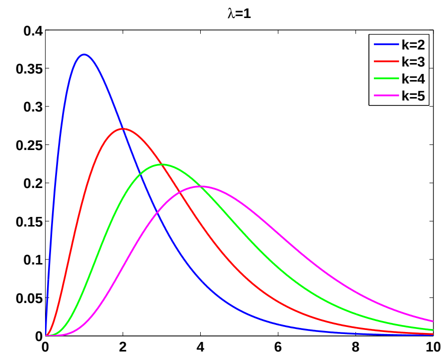
Exercise:
A transmitter sends a binary message that takes a value of either $+1$ or -1. The communications channel corrupts this message by adding to it a normal random variable with zero mean and variance σ^2. The receiver decides whether or not a -1 or 1 was sent by looking at the *sign* of the number it observes; if this number is < 0, it concludes that a -1 was sent, if it is ≥ 0, it concludes that a 1 was sent. Suppose a -1 is transmitted. What is the probability that the receiver erroneously concludes that a 1 was sent? (Your answer will of course depend on σ.)

Erlang and Gamma distributions

When we sum together k independent exponential random variables, we get an Erlang[2] random variable.

The pdf relies on two parameters: an integer k and a rate parameter $\lambda > 0$. Given these, the density function is

$$f_X(x) = \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x}, \quad x \geq 0.$$



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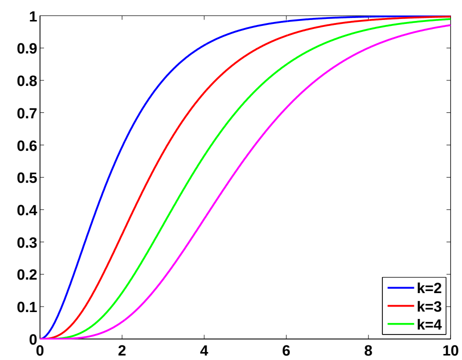
There is no closed form expression for the cdf, but we can write it as

$$F_X(x) = 1 - e^{-\lambda x} \sum_{n=0}^{k-1} \frac{\lambda^{n-1}}{n!} x^{n-1}, \quad x \geq 0.$$



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As we mentioned above, if $X_1, X_2, \ldots, X_k$ are independent with $X_i \sim \text{Exp}(\lambda)$, then

$$X = X_1 + X_2 + \cdots + X_k \quad \text{is Erlang}(k, \lambda).$$

[2] A. K. Erlang was a pioneer in network theory; he wrote down this distribution in the early 1900s while studying the number of phone call arriving at different switching stations.

From this, it is easy to see that the mean and variance are

$$E[X] = \frac{k}{\lambda}, \quad \mathrm{var}(X) = \frac{K}{\lambda^2}.$$

The pdf above can be generalized to non-integer values of k; this is called a **Gamma** distribution. For parameters $\alpha, \beta > 0$, a Gamma(α, β) random variable has pdf

$$f_X(x) = \frac{1}{\beta^{-\alpha}\Gamma(\alpha)}\, x^{\alpha-1}e^{-\beta x},$$

where $\Gamma(\cdot)$ is the so-called *Gamma function*:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\, dx.$$

It is a fact that $\Gamma(k) = (k-1)!$ for integers k, but there is no closed-form expression for non-integer values.

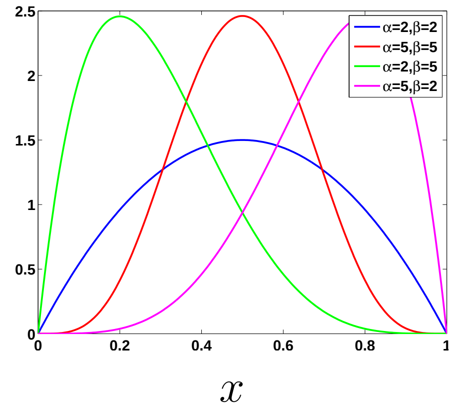
The Gamma distribution is often used to measure the accumulation of some quantity (rainfall, photons at a detector, etc) over a certain amount of time.

Beta distribution

Beta random variables takes values in the interval $[0,1]$ — they are useful for modeling things having to do with proportions or relative frequencies, and are often used when probabilities are are unknown and are are themselves treated as random.

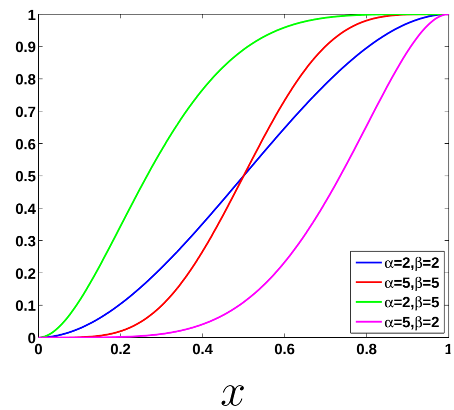
The pdf relies on two positive parameters, $\alpha, \beta > 0$, and has form

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 \le x \le 1.$$



There is no closed for expression for the cdf, and there is really no other way to write it other than

$$F_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x u^{\alpha-1}(1-u)^{\beta-1}\, du.$$



Notice that when $\alpha = \beta = 1$, the distribution is the same as Uniform($[0,1]$).

The expected value and variance of $X \sim \text{Beta}(\alpha, \beta)$ are

$$\mathrm{E}[X] = \frac{\alpha}{\alpha + \beta}, \qquad \mathrm{var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Deriving these requires some messy calculations, but the results from the central fact that

$$\int_0^1 x^p (1-x)^q \, dx = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$$

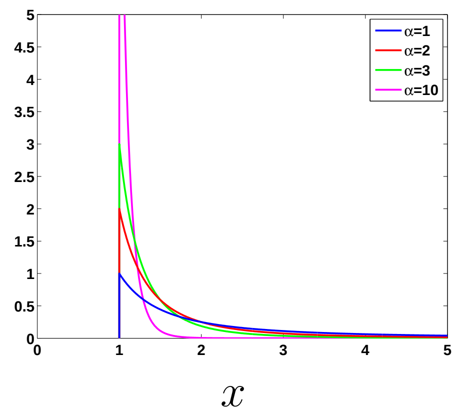
for any $p, q > 0$.

Pareto distribution

This distribution was introduced by Vilfredo Pareto in the late 1800s when studying the distribution of wealth. It is the distribution underlying the so-called "80/20 rule" — for many of the events, 80% of the effects come from 20% of the causes. This rule arises in a surprisingly diverse set of circumstances: the richest 20% of the worlds population controls approximately 80% of its income; 80% of internet traffic comes from 20% of the files being transferred; 80% of operating system crashed are caused by 20% of the bugs in the code; many companies see 80% of their sales come from 20% of their products, etc.

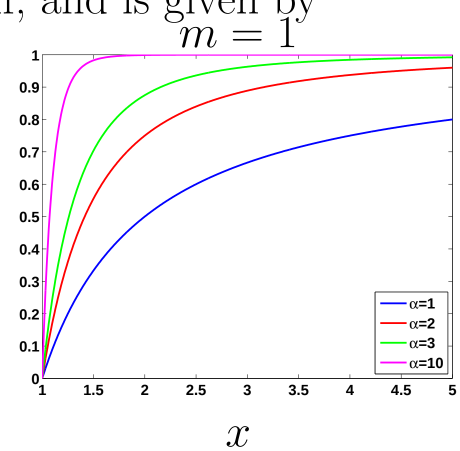
The Pareto pdf is characterized by two parameters: a minimum value m and a scale α; it has the form

$$f_X(x) = \frac{\alpha m^\alpha}{x^{\alpha+1}}, \quad x \geq m.$$



The cdf is easy to compute from the pdf, and is given by

$$F_X(x) = 1 - \left(\frac{m}{x}\right)^\alpha, \quad m \leq x.$$



The mean and variance of $X \sim \text{Pareto}(m, \alpha)$ are

$$\mathrm{E}[X] = \begin{cases} \frac{\alpha m}{\alpha-1}, & \alpha > 1, \\ \infty, & \alpha < 1, \end{cases} \qquad \mathrm{var}(X) = \begin{cases} \frac{\alpha m^2}{(\alpha-1)^2(\alpha-2)} & \alpha > 2, \\ \infty, & \alpha < 2. \end{cases}$$

As an example, we show how to calculate the expectation:

$$\begin{aligned} \mathrm{E}[X] &= \int_m^\infty x \, \frac{\alpha m^\alpha}{x^{\alpha+1}} \, dx = \alpha m^\alpha \int_m^\infty x^{-\alpha} \, dx \\ &= \alpha m^\alpha \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_m^\infty \; = \; \alpha m^\alpha \cdot \frac{m^{-\alpha+1}}{1-\alpha} \; = \; \frac{\alpha m}{1-\alpha}. \end{aligned}$$

The variance follows from a similar calculation.

To see how this pdf relates to the "80/20 rule", suppose that the random variable $X \sim \text{Pareto}(1, \alpha)$ is a model for the total value of an American's (say) assets (we are setting $m = 1$ here for simplicity; all of the calculations here will be relative rather than being valued in some currency). We will try to find a value of α so that 20% of the people have 80% of the assets.

The value of all the assets in the country is then proportional to

$$\text{Total value} = \int_1^\infty x f_X(x) \, dx = \frac{\alpha}{\alpha - 1}.$$

The total value of people with assets worth more than $x_{\min}$ is proportional to

$$\int_{x_{\min}}^\infty x f_X(x) \, dx = \frac{\alpha}{1-\alpha} \, x_{\min}^{-\alpha+1},$$

and so taking the ratio of these means that people with at least $x_{\min}$ in assets control $x_{\min}^{-\alpha+1}$ of the total assets in the country. So we need $x_{\min}^{-\alpha+1} = 0.8$.

What percentage of the people have assets worth at least $x_{\min}$? The answer to this follows immediately from the cdf:

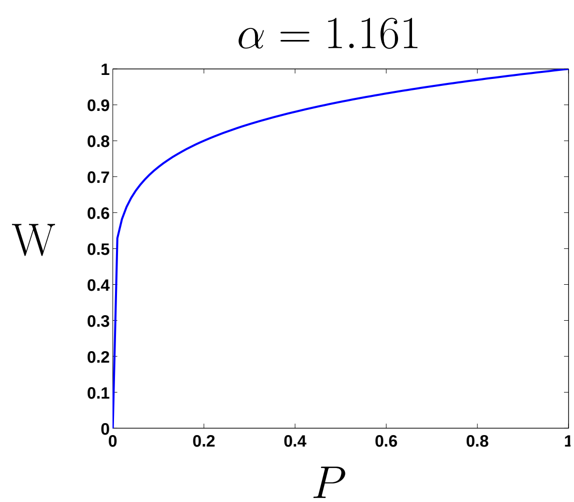
$$\mathrm{P}\left(X \geq x_{\min}\right) = 1 - F_X(x_{\min}) = x_{\min}^{-\alpha}.$$

We need this to be equal to 0.2. So we need values of $x_{\min}$ and α which satisfy:

$$\begin{aligned} x_{\min}^{-\alpha+1} &= 0.8 \\ x_{\min}^{-\alpha} &= 0.2. \end{aligned}$$

We clearly see that we need $x_{\min} = 4$, and so $\alpha = \log(5)/\log(4) = 1.161$.

For this value of α, the plot below shows the percentage of the wealth W controlled by the percentage of the people P.



In general, the wealth W controlled by the richest portion P of the population is given by

$$W(P) = P^{1-1/\alpha}.$$

As $\alpha \to 1$, a larger and larger percentage is controlled by a smaller and smaller number of people. Relationship of this type are also sometimes referred to as **power laws**.

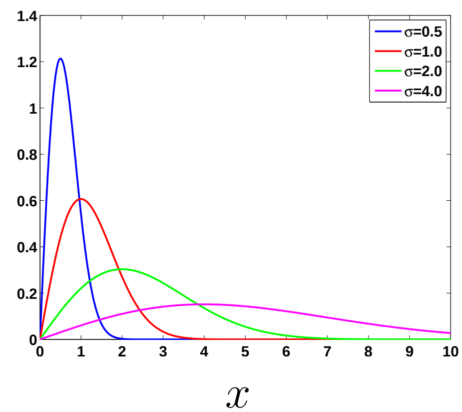
Rayleigh distribution

The Rayleigh distribution is often used to model the length of a two-dimensional vector whose coordinates are random. Suppose $U \sim \text{Normal}(0, \sigma^2)$ and $V \sim \text{Normal}(0, \sigma^2)$, and then consider the distance from the origin to the point (U, V) in the plane:

$$X = \sqrt{U^2 + V^2}.$$

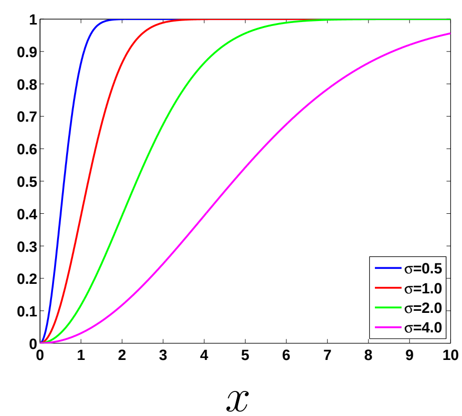
X is called a *Rayleigh* random variable, and it has pdf

$$f_X(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, \quad x \geq 0.$$



The cdf can be computed explicitly, and is given by

$$F_X(x) = 1 - e^{-x^2/2\sigma^2}, \quad x \geq 0.$$



Computing the expectation and variance of $X \sim \text{Rayleigh}(\sigma)$ is somewhat complicated, but the results are

$$\mathrm{E}[X] = \sigma\sqrt{\frac{\pi}{2}}, \qquad \mathrm{var}(X) = \frac{4-\pi}{2}\sigma^2.$$