The counting principle

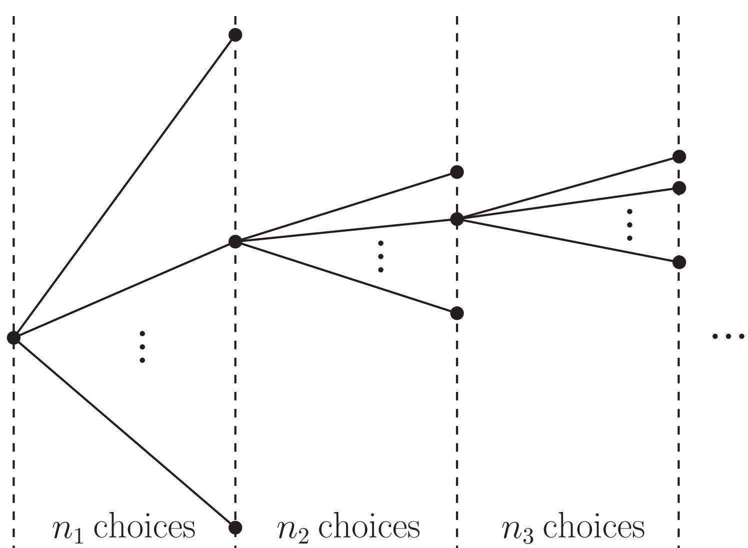
Many counting problems can be naturally broken down into multiple stages. If the outcomes at one stage do not affect the number of possibilities at the subsequent stages, we can just multiply the number of possibilities at each stage together. To formalize this, suppose that our problem has S stages, and that:

1. there are n_1 possibilities at stage 1;
2. for *every* possible result at the first stage, there are n_2 possible results at the second stage;
3. for *any* sequence of results at the first $i-1$ stages there are n_i possibilities at stage i;

then the total number of possible results is

$$n_1 \cdot n_2 \cdots n_S$$

This idea can be demonstrated graphically with a tree, where the main goal is to count the number of "leaves" on the right side.



Example. How many three letter strings are there using the letters "a"–"z"?.

$$\text{Answer: } 26 \cdot 26 \cdot 26 = 17,576.$$

Example. How many three letter strings are there where the second letter is a vowel?

$$\text{Answer: } 26 \cdot 5 \cdot 26 = 3380.$$

Example. How many three letter strings are there where the first letter is a consonant, the second letter is a vowel, and the third letter is a consonant?

$$\text{Answer: } 21 \cdot 5 \cdot 21 = 2205.$$

Example. How many three letter strings are there so that no two consecutive letters are the same?

$$\text{Answer: } 26 \cdot 25 \cdot 25 = 16,250.$$

Example. How many three letter strings are there that do not include any repeated letters?

$$\text{Answer: } 26 \cdot 25 \cdot 24 = 15,600.$$

Example. How many five letter strings are there that start with a consonant and end with "y"?

$$\text{Answer: } 21 \cdot 26 \cdot 26 \cdot 26 \cdot 1 = 369,096.$$

Sometimes we are interested in a sequence of events where the number of possibilities at the next stage depends on the outcome of the previous stages. While the goal is still to count the number of leaves, we now need to enumerate the different possible paths for getting from left to right in the tree.

Example. How many three letter strings are there that do not have consecutive vowels?

$$\text{Answer: } 5 \cdot 21 \cdot 26 + 21 \cdot 21 \cdot 26 + 21 \cdot 5 \cdot 21 = 16,401.$$

Example. How many three letter strings have two or fewer consonants?

Example. There are five parking spots arranged in a row. How many ways can two different cars park so that they are not next to one another?

Example. How many five letter strings start with "t", end with "n" and have no two consecutive letters the same?
(This is very tricky ... you will work it out on the homework.)

We can use the counting principle to derive expressions for four general types of calculations that often come up: **permutations, k-permutations, combinations and partitions**.

We can use the counting principle to derive expressions for four types of calculations.

Permutations. How many ways can I order a set of n objects?

$$\text{Answer: } n! = n(n-1)(n-2)\cdots 2\cdot 1$$

k-permutations. How many ways can I select k out of n objects and arrange them in a sequence?

$$\text{Answer: } \frac{n!}{(n-k)!}$$

Combinations. How many ways can I select k objects out of n when order does not matter?

$$\text{Answer: } \frac{n!}{k!(n-k)!}$$

Partitions. How many ways can I divide n objects into r groups where the i^{th} group has n_i members? (So $n_1 + n_2 + \cdots + n_r = n$.)

$$\text{Answer: } \frac{n!}{n_1!n_2!\cdots n_r!}$$

Permutations

How many ways can I order a set of n objects?

There are n choices for the first position, then $n-1$ choices for the second, then $n-2$ for the third, etc:

$$n(n-1)(n-2)\cdots(1) = n!$$

Example. You want to visit Nashville, Charlotte, Tampa, and Birmingham. How many ways are there to arrange your travel?

$$\text{Answer: } 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24.$$

Example. You want to visit a city in all 50 states (starting and ending in Atlanta, Georgia). How many possible ways are there to arrange your travel?

$$\text{Answer: } 49! \approx 6.083 \cdot 10^{62}$$

k-permutations

More generally, we might want to count the number of ways we can pick an **ordered subset** of size k from a larger set of size n.

As before, there are n choices for the first stage, $n-1$ for the second, and so on, but there are only k stages:

$$n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$$

Example. Suppose that 20 horses compete in the Kentucky Derby. How many ways can 1^{st}, 2^{nd}, and 3^{rd} place be awarded?

Combinations

We have a set of size n. How many subsets of size k are there?

In this case, the ordering of the subset **does not matter**. For example, all of the 2-permutations of the letters A,B,C are

$$\text{AB, BA, AC, CA, BC, CB} \qquad \text{(6 total)},$$

while all the 2-letter combinations are

$$\text{AB, AC, BC} \qquad \text{(3 total)}$$

The number of k-combinations is simply the number of k-permutations divided by the number of "duplicates" ... since we can order a set of size k in $k!$ different ways:

$$\text{“}n \text{ choose } k\text{”} = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

This quantity is also known as the *binomial coefficient*.

Example. There are 12 people on a basketball team, but only 5 people play at a time. How many different 5 man lineups are there?

$$\text{Answer: } \binom{12}{5} = \frac{12!}{7!5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2} = 11 \cdot 9 \cdot 8 = 792.$$

Example. How many binary strings of length 10 contain exactly 4 ones?

You can start to combine these ideas to count in more complex situations.

Example. In this year's graduating class, GT has n_1 ECE majors, n_2 MechE majors, n_3 BME majors and n_4 ISYE majors. The officials want to line up all of the students for commencement. How many ways can you line up the students so that all students from a given major are contiguous?

Partitions

The number of combinations $\binom{n}{k}$ can be interpreted as the number of ways we can **partition** the set into *two* subsets of sizes k and $n-k$. We can generalize the expression for the more general problem: How many ways can a set of size n be **partitioned** into r non-overlapping subsets of size $n_1, n_2, \ldots, n_r$?

(The constraint $n_1 + n_2 + \cdots + n_r = n$ is implicit.)

We answer this by forming the subsets in stages. The first subset can be formed in $\binom{n}{n_1}$ ways. For the second, $n - n_1$ elements remain, and so there are $\binom{n-n_1}{n_2}$ possibilities, etc.

So the total number of partitions is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r}$$

We can expand then simplify this as

$$\frac{n!}{n!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{r-1})!}{n_r!(n-n_1-\cdots-n_r)!}$$
$$=\frac{n!}{n_1!n_2!\cdots n_r!}.$$

This has the special notation

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\cdots n_r!},$$

which is also sometimes called the *multinomial coefficient*. Note that this is the binomial coefficient when $r = 2$.

Example (Anagrams). How many different words can be obtained by rearranging the letters in the word BANANA?

Bernoulli trials

A *Bernoulli trial* is a random experiment with two outcomes, generically labeled "success" and "failure" (or "heads"/"tails", "yes"/"no", "1"/"0", "win"/"loss", etc.) We will use p to denote the probability of success:

$$p = \mathrm{P}\left(\text{success}\right) \quad \text{and so} \quad 1 - p = \mathrm{P}\left(\text{failure}\right).$$

The phrase *Bernoulli trials* refers to a sequence of **independent** experiments of this type.

We are interested in calculating the probability of k successes in n trials.

Example. I am flipping a fair coin. What is the probability that exactly 2 out of the next 3 flips will be "heads"?

There are 8 possible outcomes and each outcome is equally likely. Since there are three ways to get a successful event, the probably of interest is

$$\mathrm{P}\left(\{\text{HHT,THH,HTH}\}\right) = \frac{3}{8}.$$

But, let's make explicit the other way we can think about this. The three successful outcomes are disjoint events, so we can write

$$\mathrm{P}\left(\{\text{HHT,THH,HTH}\}\right) = \mathrm{P}\left(\text{HHT}\right) + \mathrm{P}\left(\text{THH}\right) + \mathrm{P}\left(\text{HTH}\right).$$

For each outcome, since it is the result of independent coin tosses, we can write it using the multiplication rule: $\mathrm{P}\left(\text{HHT}\right) = \mathrm{P}\left(\text{H}\right) \cdot \mathrm{P}\left(\text{H}\right) \cdot \mathrm{P}\left(\text{T}\right) = (\frac{1}{2})^3 = \frac{1}{8} = \mathrm{P}\left(\text{THH}\right) = \mathrm{P}\left(\text{HTH}\right)$. So, the total probability is again $\mathrm{P}\left(\{\text{HHT,THH,HTH}\}\right) = \frac{3}{8}$.

In general, for any specific sequence with k successes in n trials, the probability of that sequence is $p^k(1-p)^{n-k}$.

Example. I am a 70% free throw shooter. What is the probability that I make exactly 6 out of my next 10 free throws?

The probability of any given sequence of 10 free throws with 6 makes and 4 misses is $(.7)^6(.3)^4$. To find the total probability, we simply have to add up this probability for each of these sequences. So, how many ways are there to get 6 success in 10 trials? There are $\binom{10}{6}$, giving an answer of

$$\mathrm{P}\left(\text{make exactly 6 out of 10}\right) = \binom{10}{6}(.7)^6(.3)^4 \approx 0.2001.$$

We can also calculate the probability that I make **at least** 6 out of ten:

$$\mathrm{P}\left(\text{make at least 6 out of 10}\right) = \sum_{i=6}^{10}\binom{10}{i}(.7)^i(0.3)^{10-i} \approx 0.8497.$$

In general, if the probability of success is p, then

$$\mathrm{P}\left(\text{exactly } k \text{ successes out of } n\right) = \binom{n}{k}p^k(1-p)^{n-k},$$

and

$$\mathrm{P}\left(\text{at least } k \text{ successes out of } n\right) = \sum_{i=k}^{n}\binom{n}{i}p^i(1-p)^{n-i}.$$