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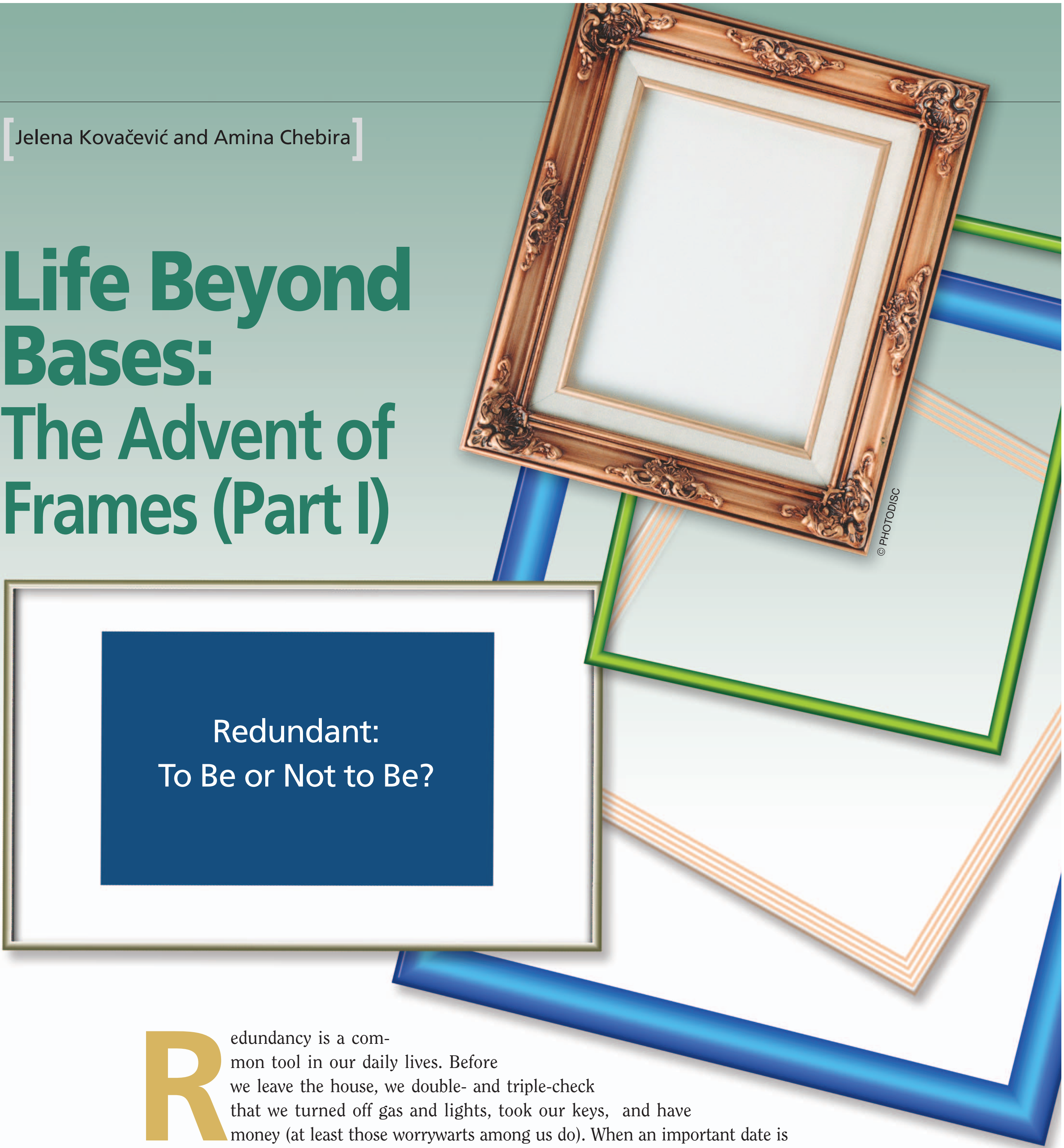
# Life Beyond Bases: The Advent of Frames (Part I)






Redundancy is a common tool in our daily lives. Before we leave the house, we double- and triple-check that we turned off gas and lights, took our keys, and have money (at least those worrywarts among us do). When an important date is coming up, we drive our loved ones crazy by confirming "just once more" they are on top of it. Of course, the reason we are doing that is to avoid a disaster by missing or forgetting something, not to drive our loved ones crazy.

The same idea of removing doubt is present in signal representations. Given a signal, we represent it in another system, typically a basis, where its characteristics are more readily apparent in the transform coefficients. However, these representations are typically nonredundant, and thus corruption or loss of transform coefficients can be serious. In comes redundancy; we build a safety net into our representation so that we can avoid those disasters. The redundant counterpart of a basis is called a *frame* [no one seems to know why they are called frames, perhaps because of the bounds in (25)?].

It is generally acknowledged (at least in the signal processing and harmonic analysis communities) that frames were born in 1952 in the paper by Duffin and Schaeffer [32]. Despite being over half a century old, frames gained popularity only in the last decade, due mostly to the work of the three wavelet pioneers—Daubechies, Grossman, and Meyer [29]. Frame-like ideas, that is, building redundancy into a signal expansion, can be found in pyramid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coding [14]; source coding [7], [8], [23], [27], [28], [37], [38], [53]; denoising [20], [31], [35], [46], [68]; robust transmission [9]–[12], [17], [36], [45], [54], [58], [63]; CDMA systems [52], [59], [66], [67]; multiantenna code design [40], [44]; segmentation [30], [50], [60]; classification [18], [50], [60]; prediction of epileptic seizures [5], [6]; restoration and enhancement [47]; motion estimation [51]; signal reconstruction [2]; coding theory [41], [55]; operator theory [1]; and quantum theory and computing [33], [57].

While frames are often associated with wavelet frames, it is important to remember that frames are more general than that. Wavelet frames possess structure; frames are redundant representations that only need to represent signals in a given space with a certain amount of redundancy. The simplest frame, appropriately named *Mercedes-Benz* (MB), is given in "The Mercedes-Benz Frame"; just have a peek at the sidebar now as we will go into more details later.

The question now is this: Why and where would one use frames? The answer is obvious: anywhere where redundancy is a must. The host of the applications mentioned above and discussed in Part II of this article [48] illustrate that richly.

Now a word about what you are reading: why an introductory article? The sources on frames are the beautiful book by Daubechies (our wavelet Bible) [27], a recent book by Christensen [19], as well as a number of classic papers [15], [26], [39], [43], among others. Although excellent material, none of the above sources offer an introduction to frames geared primarily to engineering students and those who just want an introduction into the area. Thus our emphasis in this article; this is a tutorial, rather than a comprehensive survey of the state of the field. The article comes in two parts: what you are currently reading is Part I and will cover the theoretical developments. In Part II [48], we will cover most of the known frame families as well as look into a number of applications. We will go slowly, whenever possible, using the simplest examples. Generalizations will follow naturally. We will be selective and give our personal view of frames. We will be rigorous when necessary; however, we will not insist upon it at all times. As often as possible, we will be living in the finite-dimensional world; it is rich enough to give a flavor of the basic concepts. When we do venture into the infinite-dimensional one, we will do so only using filter banks–structured expansions used in many signal processing applications.

## WHAT'S WRONG WITH BASES?

The reason we try to represent our signals in a different domain, typically, is because certain signal characteristics become obvious in that other domain facilitating various signal processing tasks. For example, we perform Fourier analysis to uncover the harmonic composition of a signal. If our signal happens to be a sum of a finite number of tones, the Fourier-domain representation will be nonzero at exactly those tones and will be zero at all other frequencies. However, if our signal is a sum of, say a pure frequency and a pulse of very short duration (for example, Dirac), the Fourier transform will be an inefficient representation; the signal energy will be, more or less, spread evenly across all frequencies. Thus, the right representation is absolutely critical if we are to perform our signal processing task effectively and efficiently.

To understand frames, it helps to go back to what we already know: bases. In this section, we review essential concepts on bases (we assume basic notions on vector spaces, inner products, norms). If you are familiar with those, you may skip this section and go directly to the frame section, which comes next. We stress that often we will forgo formal language in favor of making the material as accessible as possible. An introductory treatment is also given in [65].

When modeling a problem, one needs to identify a space of objects on which certain operations will be performed. For example, in image compression, our objects are images, while in some other tasks, our objects can be audio signals, movies, and many others. Initially, we will assume that these objects are vectors in a vector space. In this article, we consider almost exclusively finite-dimensional vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as the infinite-dimensional vector space $\ell^2(\mathbb{Z})$ (commonly used in discrete-time signal processing). By itself, a vector space will not afford much, except for the ability to add two vectors to form a new vector in the same vector space and to multiply by a scalar. To do anything meaningful, we must equip such a space with an inner product and a norm, which will allow us to "measure" things. These functions turn the vector space into an inner product space. By introducing the distance between two vectors, as the norm of the difference between those two vectors, we get a precise measurement tool and turn our inner product space into a metric space. Finally, by considering the question of completeness, that is, whether a representative set of vectors can describe every other vector from our space, we reach the Hilbert space stage, which we denote by $\mathbb{H}$. This progression allows us to do things such as measure similarity between two images by finding the distance between them, a step present in compression algorithms, systems for retrieval and matching, and many others.

We need even more: tools that will allow us to look at all the vectors in a common representation system. These tools already exist as bases in a Hilbert space. Bases are sets of vectors used to uniquely represent any vector in a given Hilbert space in terms of the basis vectors. An orthonormal basis (ONB), in particular, will allow us not only to represent vectors but to approximate them as well. This is useful when resources do not allow us to deal with the object directly but rather with its approximation only. For example, we might not have enough bits to represent $\pi$ to the tenth digit but only to the fifth one yielding 3.14159. Obviously, 3.14159 is just an approximation to 3.1415926535, which in turn is an approximation to $\pi$. Another example is compression of images. An "instant" approximation of a natural image is just its low-passed version—we get a blurry image.

### *BASES*

A subset $\Phi = \{\varphi_i\}_{i\in I}$ of a finite-dimensional vector space $\mathbb{V}$ (where $I$ is some index set) is called a *basis* for $\mathbb{V}$ if $\mathbb{V} = \mathrm{span}(\Phi)$ and the vectors in $\Phi$ are linearly independent. (Given $S \subset \mathbb{V}$, the span of $S$ is the subspace of $\mathbb{V}$ consisting of all finite linear combinations of vectors in $S$.) If $I = \{1, \ldots, n\}$, we say that $\mathbb{V}$ has dimension $n$.

A vector space $\mathbb{U}$ is *infinite dimensional* if it contains an infinite linearly independent set of vectors. If $\mathbb{U}$ is equipped with a norm, then a subset $\Phi = \{\varphi_j\}_{j\in J}$ of $\mathbb{U}$ is called a basis (or a *Schauder basis*) if for every $u$ in $\mathbb{U}$, there exist unique scalars $u_j$ such that $u = \sum_{j=1}^{\infty} u_j \varphi_j$. (Note that here we need a normed vector space because the definition implicitly uses the notion of convergence: the series converges to the vector $u$ in the norm of $\mathbb{U}$.)

### SPACES WE CONSIDER

As we already mentioned, in this article, we consider exclusively the finite-dimensional Hilbert spaces $\mathbb{H} = \mathbb{R}^n, \mathbb{C}^n$ with $I = \{1, \ldots, n\}$, as well as the infinite-dimensional space of square-summable sequences $\mathbb{H} = \ell^2(\mathbb{Z})$ with $I = \mathbb{Z}$.

$\mathbb{R}^n$ and $\mathbb{C}^n$ are the most intuitive Hilbert spaces that we deal with on a daily basis. Their dimension is $n$. For example, the complex space $\mathbb{C}^n$ is the set of all $n$-tuples $x = (x_1, \ldots, x_n)^T$, with $x_i$ in $\mathbb{C}$ (similarly for $\mathbb{R}^n$).

In discrete-time signal processing we deal almost exclusively with sequences $x$ having finite square sum or finite energy, where $x = (\ldots, x_{-1}, x_0, x_1, \ldots)$ is, in general, complex valued. Such a sequence $x$ is a vector in the Hilbert space $\ell^2(\mathbb{Z})$.

For the above spaces, the inner product is defined as

$$\langle x, y\rangle = \sum_{i\in I} x_i^* y_i,$$

while the norm is

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{i\in I} |x_i|^2}\,.$$

*A note of caution:* In the definition of a basis, we have to pay attention to the use of terms "span" and "independence" when we deal with infinite-dimensional spaces as both of these words imply *finite* linear combinations. [Many subtleties arise in infinite dimensions that are not present in finite dimensions. For instance, the infinite set $\{\delta_{i-k}\}_{k\in\mathbb{Z}}$ (here, $\delta_i = 1$ for $i = 0$ and is 0 otherwise) is a Schauder basis for $\ell^2(\mathbb{Z})$ but does not span $\ell^2(\mathbb{Z})$ because we cannot write every square-summable sequence as a finite linear combination of $\delta_i$s. For more details, we refer the reader to [42].]

### *ORTHONORMAL BASES*

A basis $\Phi = \{\varphi_i\}_{i\in I}$ where the vectors are orthonormal:

$$\langle \varphi_i, \varphi_j\rangle = \delta_{i-j}\,,$$

is called an *orthonormal basis* (ONB). In other words, an orthonormal system is called an ONB for $\mathbb{H}$, if for every $x$ in $\mathbb{H}$,

$$x = \sum_{i\in I} X_i \varphi_i\,, \tag{1}$$

for some scalars $X_i$. These scalars are called the *transform* or *expansion coefficients* of $x$ with respect to $\Phi$, and it follows from orthonormality that they are given by

$$X_i = \langle \varphi_i, x\rangle\,, \tag{2}$$

for all $i \in I$.

We now discuss a few properties of ONBs.

### PROJECTIONS

A characteristic of ONBs allowing us to approximate signals is that an orthogonal projection onto a subspace spanned by a subset of basis vectors, $\{\varphi_i\}_{i\in J}$, where $J$ is the index set of that subset is

$$Px = \sum_{i\in J} \langle \varphi_i, x\rangle \varphi_i\,, \tag{3}$$

that is, it is a sum of projections onto individual one-dimensional subspaces spanned by each $\varphi_i$. Beware that this is not true when $\{\varphi_i\}_{i\in J}$ do not form an orthonormal system.

### BESSEL'S INEQUALITY

If we have an orthonormal system of vectors $\{\varphi_i\}_{i\in J}$ in $\mathbb{V}$, then, for every $x$ in $\mathbb{V}$, the following inequality, known as Bessel's inequality, holds:

$$\sum_{i\in J} |\langle \varphi_i, x\rangle|^2 \le \|x\|^2\,.$$

### PARSEVAL'S EQUALITY

If we have an orthonormal system that is complete in $\mathbb{H}$, then we have an ONB for $\mathbb{H}$, and Bessel's relation becomes an equality, often called *Parseval's equality* (or *Plancherel's*). This is simply the norm-preserving property of ONBs. In other words

$$\|x\|^2 = \sum_{i\in I} |\langle \varphi_i,\, x\rangle|^2\,. \tag{4}$$

As an example, you might recognize this in the case of the Fourier series as

$$\|x\|^2 = \sum_{k\in\mathbb{Z}} |X_k|^2\,, \tag{5}$$

where $X_k$ are Fourier coefficients.

### LEAST-SQUARES APPROXIMATION

Suppose that we want to approximate a vector from a Hilbert space $\mathbb{H}$ by a vector lying in the (closed) subspace $S = \{\varphi_i\}_{i\in J}$.

The orthogonal projection of $x \in \mathbb{H}$ onto $S$ is given by (3). The difference vector $d = x - \hat{x}$ satisfies $d \perp S$. This approximation is best in the least-squares sense, that is, min $\|x - y\|$ for $y$ in $S$ is attained for $y = \sum_i \alpha_i \varphi_i$ with $\alpha_i = \langle \varphi_i, x \rangle$ being the expansion coefficients. In other words, the best approximation is our $\hat{x} = Px$ previously defined in (3). An immediate consequence of this result is the successive approximation property of orthogonal expansions. Call $\hat{x}^{(k)}$ the best approximation of $x$ on the subspace spanned by $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$. Then the approximation $\hat{x}^{(k+1)}$ is given by

$$\hat{x}^{(k+1)} = \hat{x}^{(k)} + \langle \varphi_{k+1}, x \rangle \, \varphi_{k+1} \, ,$$

that is, the previous approximation plus the projection along the added vector $\varphi_{k+1}$.

*A note of caution:* The successive approximation property does not hold for nonorthogonal bases. When calculating the approximation $\hat{x}^{(k+1)}$, one cannot simply add one term to the previous approximation but has to recalculate the whole approximation.

## GENERAL BASES

We are now going to relax the constraint of orthogonality and see what happens. The reasons for doing that are numerous, the most obvious one being that we have more freedom in choosing our basis vectors. For example, in $\mathbb{R}^2$, once a vector is chosen, to get an ONB, we basically have only one choice (within a sign); on the other hand, for a general basis, it is enough not to choose the second vector colinear to the first.

### EXAMPLE

As a simple example, consider the following set in $\mathbb{R}^2 : \Phi = \{\varphi_1, \varphi_2\} = \{(1, 0)^T, (\sqrt{2}/2, \sqrt{2}/2)^T\}$. We have seen how ONBs expand vectors. This is not an ONB but can we still use these two vectors to represent any real vector $x$? The answer is yes:

$$x = \langle \tilde{\varphi}_1, x \rangle \varphi_1 + \langle \tilde{\varphi}_2, x \rangle \varphi_2,$$

with $\tilde{\varphi}_1 = (1, -1)$ and $\tilde{\varphi}_2 = (0, \sqrt{2})$. Thus, we can represent any real vector with our initial pair of vectors $\Phi = \{\varphi_1, \varphi_2\}$; however, they need helpers, an extra pair of vectors $\tilde{\Phi} = \{\tilde{\varphi}_1, \tilde{\varphi}_2\}$.

So what can we say about these two couples? It is obvious that they work in concert to represent $x$. Another interesting observation is that, while not orthogonal within the couple, they are orthogonal across couples; $\varphi_1$ is orthogonal to $\tilde{\varphi}_2$ while $\varphi_2$ is orthogonal to $\tilde{\varphi}_1$. Moreover, the inner products between corresponding vectors in a couple are $\langle \varphi_i, \tilde{\varphi}_i \rangle = 1$ for $i = 1, 2$. □

In general, these *biorthogonality* relations can be compactly represented as

$$\langle \varphi_i, \tilde{\varphi}_j \rangle = \delta_{i-j} \, .$$

The representation expression can then be written as

$$x = \sum_{i \in I} \langle \tilde{\varphi}_i, x \rangle \varphi_i = \sum_{i \in I} \langle \varphi_i, x \rangle \tilde{\varphi}_i \, ,$$

that is, the roles of $\varphi_i$ and $\tilde{\varphi}_i$ are interchangeable. These two sets of vectors, $\Phi$ and $\tilde{\Phi}$, are called *biorthogonal bases* and are said to be *dual* to each other. If the dual basis $\tilde{\Phi}$ is the same as $\Phi$, we get an ONB. Thus, ONBs are self dual.

While ONBs are norm preserving, that is, they satisfy Parseval's equality, this is not true in the biorthogonal case. This is one of the reasons successive approximation does not work here. In the orthonormal case, the norm of the original vector is sliced up into pieces, each of which is the norm of the corresponding expansion coefficient (and equal to the length of the appropriate projection). Here, we know that does not work.

From the above discussion, we see that biorthogonal bases offer a larger choice, since they are less constrained than the orthonormal ones. However, this comes at the price of losing the norm-preserving property as well as the successive approximation property. This trade-off is often tackled in practice, and, depending on the problem at hand, you might decide to use either orthonormal or biorthogonal basis.

## FROM REPRESENTATIONS TO MATRICES

While we are great fans of equations, we like matrices even better as equations can be hard to parse. We strongly believe that visualizing our representations is more intuitive and helps us understand the concepts better. Thus, we rephrase our basis notions in matrix notation.

### EXAMPLE

Suppose we are given an ONB $\Phi = \{(1, -1)^T/\sqrt{2}, (1, 1)^T/\sqrt{2}\}$. Given this basis and an arbitrary vector $x$ in the plane, we might want to ask ourselves, what is this point in this new basis (new coordinate system)? We answer this question by projecting $x$ onto the new basis. Suppose that $x = (1, 0)^T$. Then, $x_{\Phi_1} = \langle \varphi_1, x \rangle = 1/\sqrt{2}$ and $x_{\Phi_2} = \langle \varphi_2, x \rangle = 1/\sqrt{2}$. Thus, in this new coordinate system, our point $(1, 0)^T$ becomes $x_\Phi = (x_{\Phi_1}, x_{\Phi_2}) = (1, 1)^T/\sqrt{2}$. It is still the same point in the plane; we only read its coordinates depending on which basis we are considering. We can express the above process of figuring out the coordinates in the new coordinate system a bit more elegantly:

$$\begin{aligned} X &= x_\Phi = \begin{pmatrix} x_{\Phi_1} \\ x_{\Phi_2} \end{pmatrix} = \begin{pmatrix} \langle \varphi_1, x \rangle \\ \langle \varphi_2, x \rangle \end{pmatrix} = \begin{pmatrix} \varphi_{11} x_1 + \varphi_{12} x_2 \\ \varphi_{21} x_1 + \varphi_{22} x_2 \end{pmatrix} \\ &= \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \Phi^* x \, , \end{aligned}$$

where * denotes Hermitian transposition. Observe that the matrix $\Phi$ describes an ONB in the real plane. (By abuse of language, we use $\Phi$ to denote both the set of vectors as well as the matrix representing those vectors.) The columns of the matrix are the basis vectors (the rows are as well), that is, the process of finding coordinates of a vector in a different coordinate system can be conveniently represented using a matrix $\Phi$ whose columns are the new basis vectors, $x_\Phi = \Phi^* x$. □

We now summarize what we learned in this example in a more general case: Any Hilbert space basis (orthonormal or biorthogonal) can be represented as a matrix having basis vectors as its columns. If the matrix is singular, it does not represent a basis.

Given that we have $X = \Phi^* x$, we can go back to $x$ by inverting $\Phi^*$ (this is why we require $\Phi$ to be nonsingular), $x = (\Phi^*)^{-1} X$. If the original basis is orthonormal, then $\Phi$ is unitary and $\Phi^{-1} = \Phi^*$. The representation formula can then be written as

$$x = \sum_{i \in I} \langle \varphi_i, x \rangle \varphi_i = \Phi \Phi^* x = \Phi^* \Phi x. \tag{6}$$

If, on the other hand, the original basis is biorthogonal, there is not much more we can say about $\Phi$. The representation formula is (the two bases $\Phi$ and $\tilde{\Phi}$ are interchangeable):

$$x = \sum_i \langle \tilde{\varphi}_i, x \rangle \varphi_i = \Phi \tilde{\Phi}^* x = \tilde{\Phi} \Phi^* x = \sum_i \langle \varphi_i, x \rangle \tilde{\varphi}_i .$$

### *SUMMARY*

To summarize what we have done in this section:

- We represented our signal in another domain to more easily extract its salient characteristics.
- Given a pair of biorthogonal bases $(\Phi, \tilde{\Phi})$, the coordinates of our signal in the new domain (or, with respect to the new basis) are given by

$$X = \tilde{\Phi}^* x, \tag{7}$$

  where $\tilde{\Phi}$ is a linear operator describing the basis change and it contains the dual basis vectors as its columns, while $X$ collects all the transform coefficients together. For $\mathbb{H} = \mathbb{R}^n, \mathbb{C}^n$, $\Phi$ is an $n \times n$ matrix; for $\mathbb{H} = \ell^2(\mathbb{Z})$, $\Phi$ is an infinite matrix. The above is called the *analysis* or *decomposition* expression.
- The *synthesis*, or, *reconstruction* is given by

$$x = \Phi X, \tag{8}$$

  where $\Phi$ is a linear operator as well, and it contains the basis vectors as its columns.
- If the expansion is into an ONB, then

$$\tilde{\Phi} = \Phi, \qquad \text{and} \qquad \Phi\Phi^* = I,$$

  that is, $\Phi$ is a unitary operator (matrix).
- If the expansion is into a pair of biorthogonal bases, then

$$\tilde{\Phi}^* = \Phi^{-1}.$$

### EXAMPLE: DFT AS AN ONB EXPANSION

The discrete Fourier transform (DFT) is ubiquitous; however, it is rarely looked upon as a signal expansion or written in matrix form. The easiest way to do that is to look at how the reconstruction is obtained:

$$x_k = \frac{1}{n} \sum_{i=0}^{n-1} X_i W_n^{ik}, \quad k = 0, \ldots, n-1, \tag{9}$$

where $W_n = e^{j2\pi/n}$ is the $n$th root of unity. In matrix notation we could write it as

$$x = \frac{1}{n} \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_n & \cdots & W_n^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & W_n^{n-1} & \cdots & W_n \end{pmatrix}}_{\Phi = \mathrm{DFT}_n} \underbrace{\begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{pmatrix}}_{X}.$$

Note that the DFT matrix defined as above is not normalized, that is $(1/n)(\mathrm{DFT}_n)(\mathrm{DFT}_n)^* = I$. If we normalized the above matrix by $1/\sqrt{n}$, the DFT would exactly implement an ONB.

The decomposition formula is usually given as

$$X_i = \sum_{k=0}^{n-1} x_k W_n^{-ik}, \quad i = 0, \ldots, n-1, \tag{10}$$

and, in matrix notation,

$$X = \mathrm{DFT}_n^* x.$$

Consider now the normalized version. In basis parlance, our basis would be $\Phi = \{\varphi_i\}_{i=0}^{n-1}$ where the basis vectors are

$$\varphi_i = \frac{1}{\sqrt{n}} \left( W_n^0, W_n^i, \ldots, W_n^{i(n-1)} \right)^T, \quad i = 0, \ldots, n-1. \tag{11}$$

Then, the expansion formula (10) can be seen as

$$X_i = \langle \varphi_i, x \rangle, \qquad i = 0, \ldots, n-1,$$

and the reconstruction formula (9) for $x = (x_0, \ldots, x_{n-1})^T$:

$$\begin{aligned} x &= \sum_{i=0}^{n-1} X_i \varphi_i = \sum_{i=0}^{n-1} \langle \varphi_i, x \rangle \varphi_i \\ &= \underbrace{\frac{1}{\sqrt{n}} \mathrm{DFT}_n}_{\Phi} \underbrace{\frac{1}{\sqrt{n}} \mathrm{DFT}_n^*}_{\Phi^*} x. \end{aligned} \tag{12}$$

### INTRODUCTION TO FRAMES

The notion of bases in finite-dimensional spaces implies that the number of representative vectors is the same as the dimension of the space. When this number is larger, we can still have a representative set of vectors, except that the vectors are no longer linearly independent and the resulting set is no longer called a basis but a frame. Frames are signal representation tools that are redundant, and since they are less constrained than bases, they are used when more flexibility in choosing a representation is needed.

In this section, we introduce frames through simple examples and consider $\mathbb{H} = \mathbb{R}^n, \mathbb{C}^n$ only. In the next section, we will define frames more formally and discuss a number of their properties. In "Finite-Dimensional Frames," we examine finite-dimensional frames in some detail. Then, in the final section, we look at the only instance of infinite-dimensional frames we discuss in this article, those in $\mathbb{H} = \ell^2(\mathbb{Z})$ implemented using filter banks.

## *GENERAL FRAMES*

### EXAMPLE

Let us take an ONB, add a vector to it, and see what happens. Suppose our system is as given in Figure 1(a), with $\Phi = \{\varphi_1, \varphi_2, \varphi_3\} = \{(1,0)^T, (0,1)^T, (1,-1)^T\}$. The first two vectors $\varphi_1, \varphi_2$ are the ones forming the ONB and the third one $\varphi_3$ was added to the ONB. What can we say about such a system?

First, it is clear that by having three vectors in $\mathbb{R}^2$, those vectors must necessarily be linearly dependent; indeed, $\varphi_3 = \varphi_1 - \varphi_2$. It is also clear that these three vectors must be able to represent every vector in $\mathbb{R}^2$ since their subset is able to do so (which also means that we could have added any other vector $\varphi_3$ to our ONB with the same result.) In other words, we know that the following is true:

$$x = \langle \varphi_1, x\rangle \varphi_1 + \langle \varphi_2, x\rangle \varphi_2 \,.$$

Nothing stops us from adding a zero to the above expression:

$$x = \langle \varphi_1, x\rangle \varphi_1 + \langle \varphi_2, x\rangle \varphi_2 + \underbrace{(\langle \varphi_1, x\rangle - \langle \varphi_1, x\rangle)(\varphi_1 - \varphi_2)}_{0} \,.$$

We now rearrange the above expression slightly to read

$$x = \langle 2\varphi_1, x\rangle \varphi_1 + \langle (-\varphi_1 + \varphi_2), x\rangle \varphi_2 + \langle -\varphi_1, x\rangle(\varphi_1 - \varphi_2) \,.$$

In the above, we can recognize $(-\varphi_1 + \varphi_2)$ as $-\varphi_3$, and the vectors inside the inner products we will call

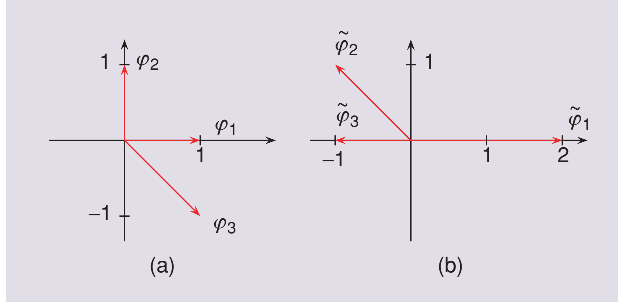
$$\tilde{\varphi}_1 = 2\varphi_1, \qquad \tilde{\varphi}_2 = -\varphi_1 + \varphi_2, \qquad \tilde{\varphi}_3 = -\varphi_1.$$

With this notation, we can rewrite the expansion as

$$x = \langle \tilde{\varphi}_1, x\rangle \varphi_1 + \langle \tilde{\varphi}_2, x\rangle \varphi_2 + \langle \tilde{\varphi}_3, x\rangle \varphi_3 = \sum_{i=1}^{3} \langle \tilde{\varphi}_i, x\rangle \varphi_i,$$

or, if we introduce matrix notation as before:

$$\Phi = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad \tilde{\Phi} = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$




**[FIG1] A pair of general frames. (a) Frame $\Phi = \{\varphi_1, \varphi_2, \varphi_3\}$. (b) Dual frame $\tilde{\Phi} = \{\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3\}$.**

and

$$x = \sum_{i=1}^{3} \langle \tilde{\varphi}_i, x\rangle \varphi_i = \Phi\tilde{\Phi}^* x \,.$$

The only difference between the above expression and the one for general bases is that matrices $\Phi$ and $\tilde{\Phi}$ are now rectangular. Figure 1 shows this example pictorially. □

We have thus shown that starting with an ONB and adding a vector, we obtained another expansion with three vectors. This expansion is reminiscent of the one for general biorthogonal bases we have seen earlier, except that the vectors involved in the expansion are now linearly dependent. This redundant set of vectors $\Phi = \{\varphi_i\}_{i\in I}$ is called a *frame* while $\tilde{\Phi} = \{\tilde{\varphi}_i\}_{i\in I}$ is called the *dual frame*. As for biorthogonal bases, these two are interchangeable, and thus, $x = \Phi\tilde{\Phi}^* x = \tilde{\Phi}\Phi^* x$.

## *TIGHT FRAMES*

Well, adding a vector worked but we ended up with an expansion that does not look very elegant. Is it possible to have frames that would somehow mimic ONBs? To do that, let us think for a moment what characterizes ONBs. It is not linear independence since that is true for biorthogonal bases as well. How about the following two facts:

- ONBs are self dual
- ONBs preserve the norm?

### EXAMPLE

Consider now the system given in "The Mercedes-Benz Frame" and the figure within, with vectors $\Phi_{\mathrm{PTF}}$ as in (16). We can easily compute $\Phi_{\mathrm{PTF}}\Phi_{\mathrm{PTF}}^* = I$ and thus, $\Phi_{\mathrm{PTF}}$ can represent any $x$ from $\mathbb{R}^2$ (real plane). Since the same set of vectors is used both for expansion and reconstruction [see (17)], $\Phi_{\mathrm{PTF}}$ is self-dual. We can think of the expansion in (17) as a generalization of an ONB except that the vectors are not linearly independent anymore. The frame of this type is called a *tight frame* (TF) and this particular one is called the *Mercedes-Benz* (MB) frame. (We will define all classes of frames in the next section.) We can normalize the lengths of all the frame vectors to one, leading to the unit-norm version of this frame given in (13). One can compare the expansion into an ONB with the expansion into a unit-norm version of the MB frame and see that the frame version has an extra scaling of 2/3 [see (14)]. When the frame is tight and all

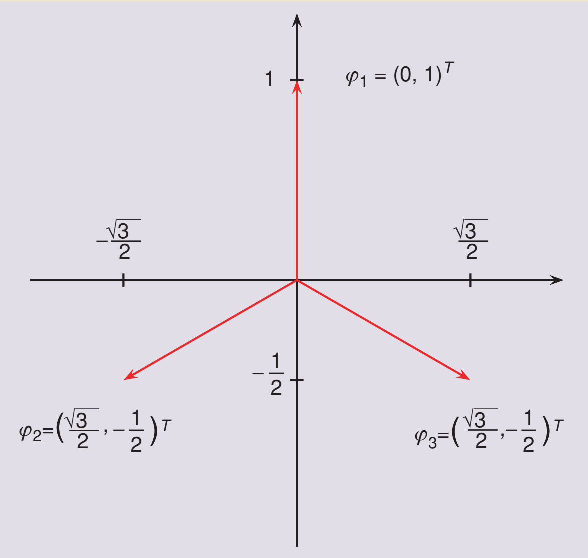
### THE MERCEDES-BENZ FRAME

The *Mercedes-Benz* (*MB*) frame is arguably the most famous frame. (MB frames are also known as Peres-Wooters states in quantum information theory [56].) It is a collection $\Phi$ of three vectors in $\mathbb{R}^2$ and is an excellent representative for many classes of frames. For example, the MB frames is the simplest harmonic TF (HTF) frame (we introduce those in Part II of this article [48]), and as such, it is also unitarily equivalent to all ENTFs in $\mathbb{R}^2$ with three vectors. Again, as an HTF frame, it can be obtained by a group operation on a single element (choose one frame vector and use rotations of $2\pi/3$).

The two equivalent versions of the MB frame are:

**UNTF Version**

The unit-norm version of the MB frame is $\Phi_{\mathrm{UNTF}} = \{\varphi_1, \varphi_2, \varphi_3\}$ (see Figure 2):




**[FIG2] Simplest unit-norm tight frame—MB frame. This is also an example of a harmonic tight frame.**

$$\Phi^*_{\mathrm{UNTF}} = \begin{pmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} \varphi_1^* \\ \varphi_2^* \\ \varphi_3^*, \end{pmatrix}, \tag{13}$$

with the corresponding expansion

$$x = \frac{2}{3}\sum_{i=1}^{3}\langle\varphi_i, x\rangle\varphi_i = \frac{2}{3}\Phi_{\mathrm{UNTF}}\Phi^*_{\mathrm{UNTF}}\, x, \tag{14}$$

and the norm

$$\|X\|^2 = \sum_{i=1}^{3}|\langle\varphi_i, x\rangle|^2 = \frac{3}{2}\|x\|^2. \tag{15}$$

**PTF Version**

The PTF version of the MB frame is $\Phi_{\mathrm{PTF}} = \{\varphi_1, \varphi_2, \varphi_3\}$, where the frame has been scaled so that $\Phi_{\mathrm{PTF}}\Phi^*_{\mathrm{PTF}} = I$

$$\Phi^*_{\mathrm{PTF}} = \sqrt{\frac{2}{3}}\Phi^*_{\mathrm{UNTF}} = \begin{pmatrix} 0 & \sqrt{2/3} \\ -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}, \tag{16}$$

and thus the expansion expression is

$$x = \sum_{i=1}^{3}\langle\varphi_i, x\rangle\varphi_i = \Phi_{\mathrm{PTF}}\Phi^*_{\mathrm{PTF}}x, \tag{17}$$

and the norm

$$\|X\|^2 = \sum_{i=1}^{3}|\langle\varphi_i, x\rangle|^2 = \|x\|^2. \tag{18}$$

**Seeding**

The PTF version of the MB frame can be obtained by projecting an ONB from a three-dimensional space (see Naimark Theorem 1):

$$\Psi = \begin{pmatrix} 0 & \sqrt{2/3} & 1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}, \tag{19}$$

using the following projection operator $P$:

$$P = \frac{1}{\sqrt{3}}\begin{pmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{pmatrix}, \tag{20}$$

that is, the MB frame seen as a collection of vectors in the three-dimensional space is $\Phi_{3D} = P\Psi$. The projection operator essentially "deletes" the last column of $\Psi$ to create the frame operator $\Phi^*$.

**Resilience to Noise**

In Part II of this article [48], we will look at the properties of this frame in the presence of additive noise; we will find that by using this frame, mean square error (MSE) per component is reduced using the MB frame. That result shows another particular property of the MB frame. Namely, among all other frames with three norm-1 frame vectors in $\mathbb{R}^2$, this particular one (and the others in the same class [36]) minimizes the MSE. With an ONB, the MSE $= \sigma^2$, while with the MB frame frame, the MSE $= (2/3)\sigma^2$.

**Resilience to Losses**

Assume now that one of the quantized coefficients is lost, for example, $\hat{X}_2$. Does our MB frame have further nice properties when it comes to losses? Note first, that even with $\varphi_2$ not present, we can still use $\varphi_1$ and $\varphi_3$ to represent any vector in $\mathbb{R}^2$. The expansion formula is just not as elegant:

$$x = \sum_{i=1,3}\langle\varphi_i, x\rangle\tilde{\varphi}_i, \tag{21}$$

with

$$\tilde{\varphi}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1 \end{pmatrix}, \qquad \tilde{\varphi}_3 = \begin{pmatrix} 2/\sqrt{3} \\ 0 \end{pmatrix}. \tag{22}$$

Calculating the MSE, we get that

$$\mathrm{MSE}_{\{2\}} = \frac{1}{2}E\|x - \hat{x}\|^2 = \frac{1}{2}E\left\| \sum_{i=1,3} w_i \tilde{\varphi}_i \right\|^2$$
$$= \frac{1}{2}\sigma^2 \sum_{i=1,3} \|\tilde{\varphi}_i\|^2 = \frac{4}{3}\sigma^2,$$

that is, twice the MSE without erasures. However, the above calculations do not tell us anything about whether there is another frame with a lower MSE. In fact, given that one element is erased, does it really matter what the original frame was? It turns out that it does. In fact, among all frames with three norm-1 frame vectors in $\mathbb{R}^2$, the MSE averaged over all possible erasures of one coefficient is minimized when the original frame is tight [36].

the vectors have unit norm as in this case, the inverse of this scaling factor denotes the redundancy of the system: we have 3/2 or 50% more vectors than needed to represent any vector in $\mathbb{R}^2$.

This discussion took care of the first question, whether we can have a self-dual frame. To check the question about norms, we compute the sum of the squared transform coefficients as in (18). We see that, indeed, this frame preserves the norm. To make the comparison to ONBs fair, again we take the unit-norm (UTNF) version of the frame and compute the sum of the squared transform coefficients as in (15). Now there is extra scaling of 3/2; this is fairly intuitive, as in the transform domain, where we have more coefficients than we started with, the energy is 3/2 times higher than in the original domain.

Thus, the TF we constructed is very similar to an ONB, with a linearly dependent set of vectors. Actually, TFs are redundant sets of vectors closest to ONBs (we will make this statement precise in "What Can Coulomb Teach Us?").

One more interesting tidbit about this particular frame; note how all its vectors have the same norm. This is not necessary for tightness but if it is true, then the frame is called an *equal-norm TF (ENTF)*. □

## *SUMMARY*

To summarize what we have done until now, assume that we are dealing with a finite-dimensional space of dimension $n$ and $m > n$ linearly dependent frame vectors:

- We represented our signal in another domain to more easily extract its salient characteristics. We did that in a redundant fashion.
- Given a pair of dual frames $(\Phi, \tilde{\Phi})$, the coordinates of our signal in the new domain (that is, with respect to the new frame) are given by

$$X = \tilde{\Phi}^* x, \tag{23}$$

where $\tilde{\Phi}$ is a rectangular $n \times m$ matrix describing the frame change and it contains the dual frame vectors as its columns, while $X$ collects all the transform coefficients together. This is called the analysis or decomposition expression.

- The synthesis, or reconstruction is given by

$$x = \Phi X, \tag{24}$$

where $\Phi$ is again a rectangular $n \times m$ matrix, and it contains frame vectors as its columns.

- If the expansion is into a *TF*, then

$$\tilde{\Phi} = \Phi, \qquad \text{and} \qquad \Phi\Phi^* = I_{n\times n}.$$

Note that, unlike for bases, $\Phi^*\Phi$ is not identity (why?).

- If the expansion is into a *general frame*, then

$$\Phi\tilde{\Phi}^* = I.$$

*A note of caution:* In frame theory, the frame change is usually denoted by $\Phi$, not $\tilde{\Phi}^*$. Given that $\Phi$ and $\tilde{\Phi}$ are interchangeable, we can use one or the other without risk of confusion. Since $\sum_{i\in I} X_i \varphi_i$ is really the expansion in terms of the basis/frame $\Phi$, it is natural to use $\Phi$ on the reconstruction side and $\tilde{\Phi}^*$ on the decomposition side.

## FRAME DEFINITIONS AND PROPERTIES

In the last section, we introduced frames through examples and developed some intuition. We now discuss frames more generally and examine a few of their properties. We formally define frames as follows: A family $\Phi = \{\varphi_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is called a *frame* if there exist two constants $0 < A \le B < \infty$, such that for all $x$ in $\mathbb{H}$,

$$A\|x\|^2 \le \sum_{i\in I} |\langle \varphi_i, x\rangle|^2 \le B\|x\|^2. \tag{25}$$

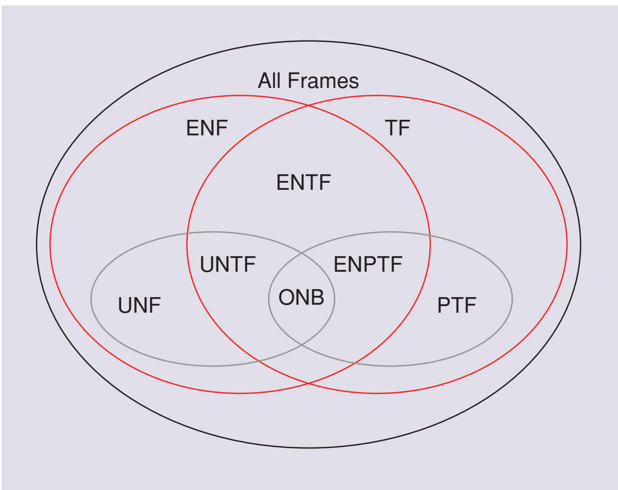
$A$, $B$ are called *frame bounds*.

The frame bounds are intimately related to the issues of stability. To have stable reconstruction, the operator mapping $x \in \ell^2(\mathbb{Z})$ into its transform coefficients $|\langle \varphi_i, x\rangle|$ has to be bounded, that is, $\sum_{i\in I} |\langle \varphi_i, x\rangle|^2$ has to be finite, achieved by the bound from above. On the other hand, no $x$ with $\|x\| > 0$ should be mapped to 0. In other words, $\|x\|^2$ and $\sum_{i\in I} |\langle \varphi_i, x\rangle|^2$ should be close. This further means that if $\sum_{i\in I} |\langle \varphi_i, x\rangle|^2 < 1$, there should exist an $\alpha < \infty$ such that $\|x\|^2 < \alpha$. For $A = 1/\alpha$, the bound from below is achieved. In summary, a numerically stable reconstruction of any $x$ from its transform coefficients is possible only if (25) is satisfied. The closer the frame bounds are, the faster and numerically better behaved reconstruction we have. In the example given in "General Frames," $A \simeq 0.8$ and $B \simeq 6.2$ and these can be computed as the smallest and largest eigenvalue of $\tilde{\Phi}\tilde{\Phi}^*$, respectively.

Frame nomenclature is far from uniform and can result in confusion. For example, frames with unit-norm frame (UNF) vectors have been called normalized frames (normalized as in all vectors normalized to norm 1, similarly to the meaning of

[TABLE 1] FRAME NOMENCLATURE.

| NAME | ABBREVIATION | DESCRIPTION | ALTERNATE NAMES |
|---|---|---|---|
| Equal-norm frame | ENF | $\|\varphi_i\| = \|\varphi_j\|$, for all $i$, $j$ | Uniform frame [49] |
| Unit-norm frame | UNF | $\|\varphi_i\| = 1$, for all $i$ | Uniform frame with norm 1 [49]<br>Uniform frame [36]<br>Normalized frame [4] |
| Tight frame | TF | $A = B$ | |
| $A$-tight frame | $A$-TF | $A = B = A$ | |
| Parseval tight frame | PTF | $A = B = 1$ | Normalized frame [3] |
| Unit-norm tight frame | UNTF | $A = B$, $\|\varphi_i\| = 1$, for all $i$ | Uniform tight frame with norm 1 [49]<br>Uniform tight frame [36]<br>Normalized tight frame [4] |




[FIG3] **Frames at a glance. ENF: Equal-norm frames. TF: Tight frames. ENTF: Equal-norm tight frames. UNF: Unit-norm frames. PTF: Parseval tight frames. UNTF: Unit-norm tight frames. ENPTF: Equal-norm Parseval tight frames. ONB: Orthonormal bases.**

normalized in ONBs), uniform, as well as uniform frames with norm 1. We now define various classes of frames. Their names, as well as alternate names under which they have been used in the literature, are given in Table 1. Figure 3 shows those same classes of frames and their relationships.

*TFs* are frames with equal frame bounds, that is, $A = B$. *Equal-norm frames* (*ENFs*) are those frames where all the elements have the same norm, $\|\varphi_i\| = \|\varphi_j\|$, for $i, j \in I$. *Unit-norm frames* (*UNFs*) are those frames where all the elements have norm 1, $\|\varphi_i\| = 1$, for $i \in I$. *A-Tight Frames* (*A-TF*) are TFs with frame bound $A$. *Parseval TFs* (*PTF*) are TFs with frame bound $A = 1$ and could also be denoted as 1-TFs.

From (25), in a TF (that is, when $A = B$), we have

$$\sum_{i \in I} |\langle \varphi_i, x \rangle|^2 = A\|x\|^2 \,. \tag{26}$$

By pulling $1/A$ into the sum, this is equivalent to

$$\sum_{i \in I} \left| \left\langle \frac{1}{\sqrt{A}} \varphi_i, x \right\rangle \right|^2 = \|x\|^2 \,, \tag{27}$$

that is, the family $\Phi = \{(1/\sqrt{A})\varphi_i\}_{i \in I}$ is a 1-TF. In other words, any TF can be rescaled to be a TF with frame bound 1, a PTF. With $A = 1$, the above looks similar to (4), Parseval's equality, thus the name PTF.

In an A-TF, $x \in \mathbb{H}$ is expanded as follows:

$$x = \frac{1}{A} \sum_{i \in I} \langle \varphi_i, x \rangle \varphi_i \,. \tag{28}$$

While this last equation resembles the expansion formula in the case of an ONB as in (1)–(2) (except for the factor $1/A$), a frame does not constitute an ONB in general. In particular, vectors may be linearly dependent and thus not form a basis. If all the vectors in a TF have unit norm, then the constant $A$ gives the redundancy ratio. For example, $A = 2$ means there are twice as many vectors than needed to cover the space. For the MB frame we discussed earlier, redundancy is 3/2, that is, we have 3/2 times more vectors than needed to represent vectors in a two-dimensional space. Note that if $A = B = 1$ (PTF), and $\|\varphi_i\| = 1$ for all $i$ (UTF), then $\Phi = \{\varphi_i\}_{i \in I}$ is an ONB (see Figure 3).

Because of the linear dependence that exists among the vectors used in the expansion, the expansion is not unique anymore. Consider $\Phi = \{\varphi_i\}_{i \in I}$ where $\sum_{i \in I} \alpha_i \varphi_i = 0$ (where not all $\alpha_i$s are zero because of linear dependence). If $x$ can be written as $x = \sum_{i \in I} X_i \varphi_i$, then one can add $\alpha_i$ to each $X_i$ without changing the decomposition. The expansion (28) is unique in the sense that it minimizes the norm of the expansion among all valid expansions. Similarly, for general frames, there exists a unique canonical dual frame, which is discussed later in this section (in the TF case, the frame and its dual are equal).

Before we proceed, we settle on notation as given in Table 2. Note that some of the concepts in the table have not been defined yet.

### FRAME OPERATORS

The *analysis operator* $\Phi^*$ maps the Hilbert space $\mathbb{H}$ into $\ell^2(I)$:

$$X_i = (\Phi^* x)_i = \langle \varphi_i, x \rangle \,, \quad i \in I \,.$$

**[TABLE 2] FRAME NOTATION.**

| SYMBOL | | EXPLANATION |
|---|---|---|
| $\mathbb{H} =$ | $\mathbb{R}^n$ | Real Hilbert space |
| | $\mathbb{C}^n$ | Complex Hilbert space |
| | $\ell^2(\mathbb{Z})$ | Space of square-summable sequences |
| $I =$ | $\{1, \ldots, m\}$ | Index set for $\mathbb{R}^n, \mathbb{C}^n$ |
| | $\mathbb{Z}$ | Index set for $\ell^2(\mathbb{Z})$ |
| When $\mathbb{H} = \mathbb{R}^n, \mathbb{C}^n$ | $n$ | Dimension of the space |
| | $m$ | Number of frame vectors |
| $\varphi_i \in \mathbb{H}$ | | Frame vector |
| $\Phi = \{\varphi_i\}_{i \in I}$ | | Frame family |
| $\Phi^*$ | | Analysis operator |
| $S =$ | $\Phi\Phi^*$ | Frame operator |
| $G =$ | $\Phi^*\Phi$ | Grammian |
| $\tilde{\varphi}_i \in \mathbb{H}$ | $S^{-1}\varphi_i$ | Dual frame vector |
| $\tilde{\Phi} = \{\tilde{\varphi}_i\}_{i \in I}$ | | Dual frame family |
| $\tilde{\Phi}^* =$ | $\Phi^* S^{-1}$ | Dual analysis operator |
| $\tilde{S} =$ | $S^{-1}$ | Dual frame operator |
| $\tilde{G} =$ | $\Phi^* S^{-2} \Phi$ | Dual Grammian |

As a matrix, the analysis operator $\Phi^*$ has rows which are the Hermitian-transposed frame vectors $\varphi_i^*$:

$$\Phi^* = \begin{pmatrix} \varphi_{11}^* & \cdots & \varphi_{1n}^* & \cdots \\ \varphi_{21}^* & \cdots & \varphi_{2n}^* & \cdots \\ \vdots & \ddots & \ddots & \cdots \\ \varphi_{m1}^* & \cdots & \varphi_{mn}^* & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

When $\mathbb{H} = \mathbb{R}^n, \mathbb{C}^n$, the above is an $m \times n$ matrix. When $\mathbb{H} = \ell^2(\mathbb{Z})$, it is an infinite matrix.

The *frame operator*, defined as $S = \Phi\Phi^*$, plays an important role. The product $G = \Phi^*\Phi$ is called the *Grammian*.

### USEFUL FRAME FACTS

When manipulating frame expressions, the frame facts given below often come in handy. It is a useful exercise for you to try to derive some of these on your own.

- For any matrix $\Phi^*$ with rows $\varphi_i^*$

$$S = \Phi\Phi^* = \sum_{i \in I} \varphi_i \varphi_i^*.$$

- If $S$ is a frame operator, then

$$\begin{aligned} Sx &= \Phi\Phi^* x = \sum_{i \in I} \langle \varphi_i, x \rangle \varphi_i, \\ \langle Sx, x \rangle &= \langle \Phi\Phi^* x, x \rangle = \langle \Phi^* x, \Phi^* x \rangle \\ &= \|\Phi^* x\|^2 = \sum_{i \in I} |\langle \varphi_i, x \rangle|^2, \\ \sum_{i \in I} \langle S\varphi_i, \varphi_i \rangle &= \sum_{i \in I} \langle \Phi^*\varphi_i, \Phi^*\varphi_i \rangle = \sum_{i,j \in I} |\langle \varphi_i, \varphi_j \rangle|^2. \end{aligned}$$

- From (25), we have that

$$AI \le S = \Phi\Phi^* \le BI$$

as well as

$$B^{-1} I \le S^{-1} \le A^{-1} I.$$

- We say that two frames $\Phi$ and $\Psi$ for $\mathbb{H}$ are *equivalent* if there is a bounded linear bijection $L$ on $\mathbb{H}$ for which $L\varphi_i = \psi_i$ for $i \in I$. [This is a mathematically simple (albeit possibly scarily sounding) way to translate the notion of "invertibility" to an infinite-dimensional Hilbert space.] Two frames $\Phi$ and $\Psi$ are *unitarily equivalent* if $L$ can be chosen to be a unitary operator. Any $A$-TF is equivalent to a PTF as $\varphi_{\text{PTF}} = (1/\sqrt{A})\varphi_{A-\text{TF}}$.
- The nonzero eigenvalues $\{\lambda_i\}_{i \in I}$, of $S = \Phi\Phi^*$ and $G = \Phi^*\Phi$ are the same. Thus

$$\operatorname{tr}(\Phi\Phi^*) = \operatorname{tr}(\Phi^*\Phi). \tag{29}$$

- A $\Phi^*$ matrix of a TF has orthonormal columns. In finite dimensions, this is equivalent to the Naimark Theorem (see next section), which says that every TF is obtained by projecting an ONB from a larger space.

### DUAL FRAME OPERATORS

The *canonical dual frame* of $\Phi$ is a frame defined as $\tilde{\Phi} = \{\tilde{\varphi}_i\}_{i \in I}$, where

$$\tilde{\varphi}_i = S^{-1}\varphi_i, \quad i \in I. \tag{30}$$

Noting that $\tilde{\varphi}_i^* = \varphi_i^* S^{-1}$ and stacking $\tilde{\varphi}_1^*, \tilde{\varphi}_2^*, \ldots$, in a matrix, the analysis frame operator associated with $\tilde{\Phi}$ is

$$\tilde{\Phi}^* = \Phi^* S^{-1},$$

while its frame operator is $S^{-1}$, with $B^{-1}$ and $A^{-1}$ its frame bounds. Since

$$\Phi\tilde{\Phi}^* = \underbrace{\Phi\Phi^*}_{S} S^{-1} = I,$$

then

$$x = \sum_{i \in I} \langle \tilde{\varphi}_i, x \rangle \varphi_i = \Phi\tilde{\Phi}^* x = \tilde{\Phi}\Phi^* x.$$

### FINITE-DIMENSIONAL FRAMES

We now consider finite-dimensional frames, that is, when $\mathbb{H} = \mathbb{R}^n, \mathbb{C}^n$, and examine few of their properties.

For example, for an ENTF with norm-$a$ vectors, since $S = \Phi\Phi^* = AI_{n \times n}$,

$$\operatorname{tr}(S) = \sum_{j=1}^{n} \lambda_j = nA, \tag{31}$$

where $\lambda_j$ are the eigenvalues of $S = \Phi\Phi^*$. On the other hand, because of (29)

**[TABLE 3] SUMMARY OF PROPERTIES FOR VARIOUS CLASSES OF FRAMES. ALL TRACE IDENTITIES ARE GIVEN FOR $\mathbb{H} = \mathbb{R}^n, \mathbb{C}^n$.**

| FRAME | CONSTRAINTS | PROPERTIES |
|---|---|---|
| General | $\{\varphi_i\}_{i\in I}$ is a Riesz basis for $\mathbb{H}$ | $A\|x\|^2 \le \sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 \le B\|x\|^2$ <br> $AI \le S \le BI$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2$ |
| ENF | $\|\varphi_i\| = \|\varphi_j\| = a$ for all $i$ and $j$ | $A\|x\|^2 \le \sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 \le B\|x\|^2$ <br> $AI \le S \le BI$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2 = ma^2$ |
| TF | $A = B$ | $\sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 = A\|x\|^2$ <br> $S = AI$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = nA = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2$ |
| PTF | $A = B = 1$ | $\sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 = \|x\|^2$ <br> $S = I$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = n = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2$ |
| ENTF | $A = B$ <br> $\|\varphi_i\| = \|\varphi_j\| = a$ for all $i$ and $j$ | $\sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 = A\|x\|^2$ <br> $S = AI$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = nA = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2 = ma^2$ <br> $A = (m/n)a^2$ |
| UNTF | $A = B$ <br> $\|\varphi_i\| = 1$ for all $i$ | $\sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 = A\|x\|^2$ <br> $S = AI$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = nA = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2 = m$ <br> $A = m/n$ |
| ENPTF | $A = B = 1$ <br> $\|\varphi_i\| = \|\varphi_j\| = a$ for all $i$ and $j$ | $\sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 = \|x\|^2$ <br> $S = I$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = n = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2 = ma^2$ <br> $a = \sqrt{n/m}$ |
| UNPTF <br> ⇔ <br> ONB | $A = B = 1$ <br> $\|\varphi_i\| = 1$ <br> for all $i$ | $\sum_{i\in I} \|\langle \varphi_i, x\rangle\|^2 = \|x\|^2$ <br> $S = I$ <br> $\mathrm{tr}(S) = \sum_{j=1}^n \lambda_j = n = \mathrm{tr}(G) = \sum_{i=1}^m \|\varphi_i\|^2 = m$ <br> $n = m$ |

$$\mathrm{tr}(S) = \mathrm{tr}(G) = \sum_{i=1}^{m} \|\varphi_i\|^2 = ma^2 . \quad (32)$$

Combining (31) and (32), we get

$$A = \frac{m}{n} a^2 . \quad (33)$$

Then, for a UNTF, that is, when $a = 1$, (23) yields the redundancy ratio:

$$A = \frac{m}{n} .$$

Recall that for the MB frame, $A = 3/2$.

These and other trace identities for all frames classes are given in Table 3.

### *NAIMARK THEOREM*

The following theorem tells us that every PTF can be realized as a projection of an ONB from a larger space. The theorem has been rediscovered by several people in the past decade: The first author heard it from Daubechies in the mid 1990s. Han and Larson rediscovered it in [39]; they came up with the idea that a frame could be obtained by compressing a basis in a larger space and that the process is reversible. Finally, it was Šoljanin [57] who pointed out to the first author that this is, in fact, Naimark's theorem, which has been widely known in operator algebra and used in quantum information theory. In this article, we consider only the finite-dimensional instantiation of the theorem.

#### THEOREM 1 (NAIMARK [1], HAN AND LARSON [39])

*A set $\Phi = \{\varphi_i\}_{i\in I}$ in a Hilbert space $\mathbb{H}$ is a PTF for $\mathbb{H}$ if and only if there is a larger Hilbert space $\mathbb{K}$, $\mathbb{H} \subset \mathbb{K}$, and an ONB $\{e_i\}_{i\in I}$ for $\mathbb{K}$ so that the orthogonal projection $P$ of $\mathbb{K}$ onto $\mathbb{H}$ satisfies: $Pe_i = \varphi_i$, for all $i \in I$.*

While the above theorem specifies how all TFs are obtained, the same is true in general, that is, any frame can be obtained by projecting a biorthogonal basis from a larger space [39] (we are talking here about finite dimensions only). We will call this process *seeding* and will say that a frame $\Phi$ is obtained by seeding from a basis $\Psi$ by deleting a suitable set of columns of $\Psi$ [54]. We denote this as

$$\Phi^* = \Psi[J] ,$$

where $J \subset \{1, \ldots, m\}$ is the index set of the retained columns.

We can now reinterpret the PTF identity $\Phi\Phi^* = I$ (see Table 3): It says that the columns of $\Phi^*$ are orthonormal. In view of the above theorem, this makes a lot of sense as that frame was obtained by deleting columns from an ONB from a larger space.

For example, for the MB frame given in the sidebar "The Mercedes-Benz Frame," the three-dimensional ONB from which it is obtained is given in (19) and the projection operator in (20). The MB frame obtained is in its PTF version given in (16).

### WHAT CAN COULOMB TEACH US?

As the ONBs have specific characteristics highly prized among bases, the same distinction belongs to TFs among all frames. As such, they have been studied extensively but only recently have Benedetto and Fickus [4] formally shown why TFs and ONBs indeed belong together. In their work, they characterized all UNTFs, while in [16], the authors did the same for nonequal norm TFs.

To characterize UNTFs, as a starting point, the authors looked at *harmonic TFs* (we will introduce those in Part II of this article [48]), obtained by taking $m$th roots of unity in $\mathbb{R}^n$. These lead to regular arrangement of points on a circle. An example is the MB frame from Figure 2. Trying to generalize the notion of geometric regularity to three dimensions, they looked at vertices of regular polyhedra but came short as there are only five such Platonic solids. Considering other sets of high symmetry such as the "soccer ball" (a truncated icosahedron), they found that all these proved to be UNTFs.

As the geometric intuition could lead them only so far, the authors in [4] refocused their attention on the equidistribution properties of these highly symmetric objects and thought of the notion of equilibrium. To formalize that notion, they turned to classical physics and considered the example of $m$ electrons on a conductive spherical shell. In the absence of external forces, electrons move according to the Coulomb force law until they reach the state of minimum potential energy (though that minimum might only be a local minimum leading to an unstable equilibrium). The intuition developed through this example lead them to the final result.

The authors tried to replicate the physical world for the simplest UNTFs—ONBs and thought of what kind of equilibrium they possessed. Clearly, whichever "force" acts on the vectors in an ONB, it tries to promote orthogonality. For example, the Coulomb force would not keep the ONB in a state of equilibrium. (Think $n = 2$, the Coulomb force would position the two vectors to be colinear of opposite sign.) Thus, the authors set to find another such force: the orthogonality-promoting one. This force should be repulsive if vectors form an acute angle, while it should be attractive if they form an obtuse angle. Since points are restricted to move only on the circle (unit-norm constraint), one can consider only the tangential component of the force. Even if vectors do not all have equal norm, $\|\varphi_i\| = a_i$, for $i \in I$, one can define the *frame force FF* on the whole space:

$$\begin{aligned} FF(\varphi_i, \varphi_j) &= 2\langle \varphi_i, \varphi_j \rangle (\varphi_i - \varphi_j) \\ &= \left(a_i^2 + a_j^2 - \|\varphi_i - \varphi_j\|^2\right)(\varphi_i - \varphi_j). \end{aligned}$$

Following the physical trail, one can now define the potential between two points as:

$$P(\varphi_i, \varphi_j) = p(\|\varphi_i - \varphi_j\|).$$

This is found by using $p'(x) = -xf(x)$, where $f(x)$ is the scalar part of the frame force and $p(x)$ is obtained by integrating the above and evaluating at $\|\varphi_i - \varphi_j\|^2$. After some manipulations, the result is

$$P(\varphi_i, \varphi_j) = \langle \varphi_i, \varphi_j \rangle^2 - \frac{1}{4}\left(a_i^2 + a_j^2\right)^2.$$

Then, the total potential contained within a sequence is

$$TP(\Phi = \{\varphi_i\}_{i \in I}) = \sum_{i,j \in I, i \neq j} |\langle \varphi_i, \varphi_j \rangle|^2 - \frac{1}{4}\sum_{i,j}\left(a_i^2 + a_j^2\right)^2.$$

Physically, we can interpret the total potential as follows: Given two sequences of points, the difference in potentials between these two sequences is the energy needed to move the points from one configuration to the other. As potential energy is defined in terms of differences, it is unique up to additive constants and thus we can neglect the constants and add the diagonal terms to obtain the final expression for the frame potential:

$$FP(\Phi = \{\varphi_i\}_{i \in I}) = \sum_{i,j \in I} |\langle \varphi_i, \varphi_j \rangle|^2. \tag{34}$$

Thus, what we are looking for are those sequences in equilibrium under the frame force, and these will be minimizers of the frame potential.

For UNTFs, Benedetto and Fickus discovered the following:

#### THEOREM 2 [4]

*Given $\Phi = \{\varphi_i\}_{i=1}^m$, with $\varphi_i \in \mathbb{H}^n$, consider the frame potential given in (34). Then:*

*1) Every local minimizer of the frame potential is also a global minimizer.*

*2) If $m \leq n$, the minimum value of the frame potential is*

$$FP = n,$$

*and the minimizers are precisely the orthonormal sequences in $\mathbb{R}^n$.*

*3) If $m \geq n$, the minimum value of the frame potential is*

$$FP = \frac{m^2}{n},$$

*and the minimizers are precisely the UNTFs for $\mathbb{R}^n$.*

This result tells us a few things:

1) Minimizing the frame potential amounts to finding sequences whose elements are "as orthogonal" to each other as possible.

2) UNTFs are a natural extension of ONBs, that is, the theorem formalizes the intuitive notion that UNTFs are a generalization of ONBs.

3) Both ONBs and UNTFs are results of the minimization of the frame potential, with different parameters (number of elements equal/larger then the dimension of the space).

What happens if points live on different spheres, $\varphi_i = a_i$ (vectors are not of equal norm)? Again, we can try to minimize the frame potential. Since now points live on spheres of different radii, it is intuitive that stronger points (with a larger norm) will be able to be "more orthogonal" than the weaker ones. If the strongest point is strong enough, it grabs a dimension to itself and leaves the others to squabble over what is left. We start all over with the second one and continue until those points left have to share. This is governed by the Fundamental Inequality given in (36), which says that if no point is stronger than the rest they immediately have to share, leading to TFs. In other words, when $m$ points in an $n$-dimensional space are in equilibrium, we can divide those points into two sets: a) Those "stronger" than the rest. These $(i_0 - 1)$ points get a dimension each and are thus orthogonal to each other. b) Those "weaker" than the rest. These points get the rest of the $(n - i_0 + 1)$ dimensions and form a TF for their span. If no point is the "strongest," they all have to share the space leading to a TF, as per the Fundamental Inequality. This discussion is summarized in Theorem 3.

### THEOREM 3 ([16])

*Given a sequence $\{a_i = \|\varphi_i\|\}_{i=1}^m$ in $\mathbb{R}$, such that $a_1 \geq \cdots \geq a_m \geq 0$, and any $n \leq m$, let $i_0$ denote the smallest index $i$ for which*

$$(n - i)a_i^2 \leq \sum_{j=i+1}^{m} a_j^2, \tag{35}$$

*holds. Then, any local minimizer of the frame potential is of the form*

$$\Phi = \{\varphi_i\}_{i=1}^m = \{\varphi_i\}_{i=1}^{i_0-1} \cup \{\varphi_i\}_{i=i_0}^m ,$$

*where $\Phi_o = \{\varphi_i\}_{i=1}^{i_0-1}$ is an orthogonal set and $\Phi_f = \{\varphi_i\}_{i=i_0}^m$ forms a TF for the orthogonal complement of the span of $\Phi_o$.*

The immediate corollary is the *Fundamental Inequality* we talked about, which says that if no point is stronger than the rest, the vectors have to share the space, leading to a TF when

$$\max_{i \in I} a_i^2 \leq \frac{1}{n} \sum_{i \in I} a_i^2 . \tag{36}$$

The frame potential defined in (34) is a concept introduced by Benedetto and Fickus, and it proved immediately useful. For example, it was used in [17] to show how to packetize coefficients in transmission with erasures to minimize the error of reconstruction. A decade before [4], Massey and Mittelholzer [52] used the frame potential (albeit not calling it the frame potential) as the total user interference in code-division multiple access (CDMA) systems. Minimizing that interference lead to the spreading sequences (of length $n$) being a TF (minimum of the Welch's bound). This will be discussed in more detail in Part II of this article [48].

### *DESIGN CONSTRAINTS: WHAT MIGHT WE ASK OF A FRAME?*

When designing a frame, particularly if we have a specific application in mind, it is useful to list potential requirements we might impose on our frame.

- Tightness (T): This is a very common requirement. Typically, tightness is imposed when we do need to reconstruct. Since TFs do not require inversion of matrices, they seem a natural choice.
- Equal norm (EN): In the real world, the squared norm of a vector is usually associated with power. Thus, in situations where equal-power signals are desirable, equal norm is a must.
- Maximum robustness (MR): We call a frame maximally robust to erasures (MR), if every $n \times n$ submatrix of $\Phi^*$ is invertible. This requirement arose in using frames for robust transmission [36] and will be discussed in more detail in Part II of this article [48].
- Equiangularity (EA): This is a geometrically intuitive requirement. We ask for angles between any two vectors to be the same. There are many more (tight) frames than those which are equiangular, so this leads to a very particular class of frames. These are discussed in more detail in Part II of this article [48].
- Symmetry (S): Symmetries in a frame are typically connected to its geometric configuration. Harmonic and equiangular frames are good examples. See the work of Vale and Waldron [62] for details.

*Invariance of Frame Properties:* When designing frames, it is useful to know which transformations will not destroy properties our frame already possesses. For that reason, we list below a number of frame invariance properties [54].

Let $\Phi$ be a frame. In all matrix products below, we assume the sizes to be compatible.

- $A\Phi B$ is a frame for any invertible matrices $A$, $B$.
- If $\Phi$ is TF/UNTF, then $aU\Phi V$ is TF/UNTF for any unitary matrices $U$, $V$ and $a \neq 0$.
- If $\Phi$ is EN, then $aD\Phi U$ is EN for any diagonal unitary matrix $D$, unitary matrix $U$, and $a \neq 0$.
- If $\Phi$ is MR, then $D\Phi A$ is MR for any invertible diagonal matrix $D$ and any invertible matrix $A$.
- If $\Phi$ is UNTF MR, then $D\Phi U$ is UNTF MR for any unitary diagonal matrix $D$ and any unitary matrix $U$.

### INFINITE-DIMENSIONAL FRAMES VIA FILTER BANKS

We now consider the only infinite-dimensional class of frames discussed in this article, those implemented by filter banks, the reason being that these are frames used in applications and our only link to the real world. The vectors (signals) live in the infinite-dimensional Hilbert space $\mathbb{H} = \ell^2(\mathbb{Z})$. An in-depth treatment of filter banks is given in [61], while a more expansion-oriented approach is followed in [64] and [65].

### *FILTER BANK VIEW OF BASES*

As we have done earlier in the article, we will first examine how filter banks implement bases and then move onto frames.

We have seen that we want to find representations or matrices $\Phi$ and $\tilde{\Phi}$ such that $\Phi\tilde{\Phi}^* = I$. As of now, we have presented a generic matrix $\Phi$, but how do we choose it? Of course, we want it to have some structure and lead to efficient representations of signals. Since now we are dealing with infinite-dimensional matrices, this might be easier said than done.

### EXAMPLE

Suppose we have the following two vectors: $\varphi_0 = (\cdots, 0, 1, 1, 0, \cdots)^T/\sqrt{2}$, and $\varphi_1 = (\cdots, 0, 1, -1, 0, \cdots)^T/\sqrt{2}$. These vectors form a basis for their span, that is, they can represent any two-dimensional vector, but not any vector in $\ell^2(\mathbb{Z})$. Now, define $\tau_i$ as a shift by $i$, that is, if $x = (\ldots, x_{-1}, x_0, x_1, \ldots)^T \in \ell^2(\mathbb{Z})$, then $\tau_i x = (\ldots, x_{-i-1}, x_{-i}, x_{-i+1}, \ldots)^T$ is its shifted version by $i$. Let us form the following matrix:

$$\Phi^* = \begin{pmatrix} \vdots \\ (\tau_{-2}\varphi_0)^* \\ (\tau_{-2}\varphi_1)^* \\ (\varphi_0)^* \\ (\varphi_1)^* \\ (\tau_2\varphi_0)^* \\ (\tau_2\varphi_1)^* \\ \vdots \end{pmatrix},$$

that is, the columns of $\Phi$ are the two vectors $\varphi_0, \varphi_1$ and all their even shifts:

$$\Phi^* = \frac{1}{\sqrt{2}} \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & -1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

From this expression we can see that $\Phi$ is block diagonal. If we denote by

$$\Phi_0^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{37}$$

then, $\Phi$ can be written as

$$\Phi^* = \begin{pmatrix} \ddots & & & & \\ & \Phi_0^* & & & \\ & & \Phi_0^* & & \\ & & & \Phi_0^* & \\ & & & & \ddots \end{pmatrix}.$$

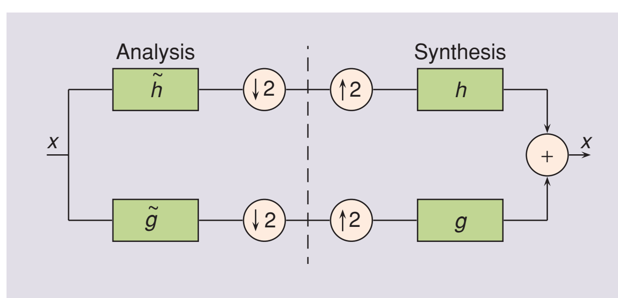
This is known as the Haar transform and is an example of a block transform. The matrix $\Phi^*$ above is unitary and corresponds to an orthonormal expansion. The basis $\Phi$ is given by $\Phi = \{\varphi_{2i}, \varphi_{2i+1}\}_{i\in\mathbb{Z}} = \{\tau_{2i}\varphi_0, \tau_{2i}\varphi_1\}_{i\in\mathbb{Z}}$. Therefore, any $x \in \ell^2(\mathbb{Z})$ can be represented using the Haar ONB as

$$x = \Phi\Phi^* x = \sum_{i\in\mathbb{Z}} \langle \varphi_i, x\rangle \varphi_i,$$

and can be implemented using the two-channel filter bank shown in Figure 4. The decomposition is implemented using the *analysis* filter bank, while the reconstruction is implemented using the synthesis filter bank (we will make this more precise shortly). □

In general, in such a filter bank, one branch is a low-pass channel that captures the coarse representation of the input signal and the other branch is a high-pass channel that captures a complementary, detailed representation. The input into the filter bank is a square-summable infinite sequence $x \in \ell^2(\mathbb{Z})$. Assuming that the filter length $l = 2$, the two analysis filters act on two samples at a time and then, due to downsampling by two, the same filters act on the following two samples. In other words, there is no overlap. On the synthesis side, the reverse is true. This is an example of a block transform. Iterating this block (the two-channel FB) on either channel or both leads to various signal transforms, each of which is adapted to a class of signals with different energy concentrations in time and in frequency (this is usually referred to as "tiling of the time-frequency plane").

So how exactly is the filter bank related to the matrix $\Phi$? In our discussion above and the Haar example, we assumed that the filter length is equal to the shift. This is not true in general, and now, we lift that restriction and allow filters to be of arbitrary length $l$ (without loss of generality, we will assume that filters are causal, that is, they are nonzero only for positive indices). However, we do leave the restriction that the filters are finitely supported, that is, they are finite impulse response (FIR) filters. (IIR filters also fit in this framework, we concentrate on FIR only for simplicity. Moreover, this restriction makes all the operators bounded and all the series converge.) Consider an inner product between two sequences $x$ and $y$ (on the left), and filtering a sequence $x$ by a filter $f$ and having the output at time $k$ (on the right):




**[FIG4] Two-channel filter bank with downsampling by two.**

$$\langle x, y\rangle = \sum_{i\in\mathbb{Z}} x_i^* y_i \qquad\qquad x * f = \sum_{i\in\mathbb{Z}} x_i^* f_{k-i}\,.$$

By comparing the above two expressions, we see that we could express filtering a sequence $x$ by a filter $f$ and having the output at time $k$ as

$$\sum_{i\in\mathbb{Z}} x_i^* f_{k-i} = \langle f_{k-i}, x_i\rangle\,.$$

Thus, to express the analysis part of the filter bank, we can do the following:

$$X = \begin{pmatrix} \vdots \\ X_0 \\ X_1 \\ X_2 \\ X_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \langle\tilde{\varphi}_0, x\rangle \\ \langle\tilde{\varphi}_1, x\rangle \\ \langle\tilde{\varphi}_2, x\rangle \\ \langle\tilde{\varphi}_3, x\rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \langle\tilde{g}_{-i}, x_i\rangle \\ \langle\tilde{h}_{-i}, x_i\rangle \\ \langle\tilde{g}_{2-i}, x_i\rangle \\ \langle\tilde{h}_{2-i}, x_i\rangle \\ \vdots \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \tilde{g}_3 & \tilde{g}_2 & \tilde{g}_1 & \tilde{g}_0 & \cdots \\ \cdots & \tilde{h}_3 & \tilde{h}_2 & \tilde{h}_1 & \tilde{h}_0 & \cdots \\ \cdots & \tilde{g}_5 & \tilde{g}_4 & \tilde{g}_3 & \tilde{g}_2 & \cdots \\ \cdots & \tilde{h}_5 & \tilde{h}_4 & \tilde{h}_3 & \tilde{h}_2 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{\tilde{\Phi}^*} \underbrace{\begin{pmatrix} \vdots \\ x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}}_{x} = \tilde{\Phi}^* x\,.$$

Similarly, the reconstruction part can be expressed as

$$x = \underbrace{\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & g_2 & h_2 & g_0 & h_0 & 0 & 0 & \cdots \\ \cdots & g_3 & h_3 & g_1 & h_1 & 0 & 0 & \cdots \\ \cdots & g_4 & h_4 & g_2 & h_2 & g_0 & h_0 & \cdots \\ \cdots & g_5 & h_5 & g_3 & h_3 & g_1 & h_1 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{\Phi} \underbrace{\begin{pmatrix} \vdots \\ X_0 \\ X_1 \\ X_2 \\ X_3 \\ \vdots \end{pmatrix}}_{X}$$

$$= (\cdots \quad \tau_{-2}g \quad \tau_{-2}h \quad g \quad h \quad \tau_2 g \quad \tau_2 h \quad \cdots)\, X$$

$$= \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \Phi_1 & \Phi_0 & 0 & \cdots \\ \cdots & \Phi_2 & \Phi_1 & \Phi_0 & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} X = \Phi X\,, \tag{38}$$

where $\Phi_i$ are $n \times m$ matrices with $n$ being the shift and $m$ the number of channels/filters in the filter bank. The matrices are formed by taking the $i$th block of $n$ coefficients from each of the $m$ filters. Here $n = 2$ and $m = 2$.

Recall again, we have assumed our filters to be causal. From above, we can conclude the following:

- The basis is $\Phi = \{\tau_{2i}\varphi_0, \tau_{2i}\varphi_1\}_{i\in\mathbb{Z}} = \{\tau_{2i}g, \tau_{2i}h\}_{i\in\mathbb{Z}}$. In other words, the impulse responses of the template filters $g$ and $h$ and their even shifts form the basis $\Phi$ (they are the columns of $\Phi$).
- The dual basis is $\tilde{\Phi} = \{\tau_{2i}\tilde{\varphi}_0, \tau_{2i}\tilde{\varphi}_1\}_{i\in\mathbb{Z}} = \{\tau_{2i}\tilde{g}, \tau_{2i}\tilde{h}\}_{i\in\mathbb{Z}}$. In other words, the impulse responses of the template filters $\tilde{g}$ and $\tilde{h}$ and their even shifts form the basis $\tilde{\Phi}$ (they are the columns of $\tilde{\Phi}$).
- When $\tilde{\Phi} = \Phi$, the basis is orthonormal. In that case, $\tilde{g}_i = g_{-i}$, that is, the impulse responses of the analysis filters are time-reversed impulse responses of synthesis filters.
- The even shifts appear because of down/upsampling by two.
- When the filters are of length $l = 2$ ($l = n$ in general), $\Phi^*$ or $\tilde{\Phi}^*$ contain only one block, $\Phi_0^*$ or $\tilde{\Phi}_0^*$, along the diagonal, making it a block-diagonal matrix (as in the Haar transform). The effect of this is that the input is processed in nonoverlapping pieces of length two. Effectively, this is equivalent to dealing with bases in the two-dimensional space.
- We discussed here a specific case with two template filters and shifts by two. In filter bank parlance, we discussed two-channel filter banks with sampling by two. (By sampling, we mean the two sampling operations, downsampling and upsampling.) Of course, more general options are possible and one can have $m$-channel filter banks with sampling by $m$. We then have $m$ template filters (basis vectors) from which all the basis vectors are obtained by shifts by multiples of $m$. The blocks $\Phi_i^*$ then become of size $m \times m$. Again, if filters are of length $l = m$, this leads to the block-diagonal $\Phi^*$ and, effectively, finite-dimensional bases.

### *z*-DOMAIN VIEW OF SIGNAL PROCESSING

Historically, the above, basis-centric view of filter banks came very recently. Initially, when the filter banks were developed to deal with speech coding [22], [34], the analysis was done in $z$-domain (for easier algebraic manipulation). In particular, $z$-transform comes in handy when we have to deal with shift-varying systems such as filter banks. Shift variance is introduced into the system due to downsamplers (or shifts). A tool used to transform a filter bank from a single-input, single-output (SISO) linear periodically shift-variant system into a multiple-input, multiple-output (MIMO) linear shift-invariant systems is called the *polyphase transform*.

For $i = 0, \ldots, m-1$, for the $i$th synthesis filter (template basis vector), $(\varphi_{i0}(z), \ldots, \varphi_{im-1}(z))^T$ is called the *polyphase representation of the ith synthesis filter* where

$$\varphi_{ik}(z) = \sum_{p\in\mathbb{Z}} \varphi_{i,mp+k} z^{-p}\,, \tag{39}$$

are the *polyphase components* for $i, k = 0, \ldots, m-1$. To relate $\varphi_{ik}(z)$ to a time-domain object, note that it is the discrete-time Fourier transform of the template basis vector $\varphi_i$ obtained by retaining only the indices congruent to $k$ modulo $m$. Then $\Phi_p(z)$ is the corresponding $m \times m$ synthesis polyphase matrix with elements $\varphi_{ik}(z)$. In other words, a polyphase decomposition is a decomposition into $m$ subsequences modulo $m$. We can do the same on the analysis side, leading to the polyphase matrix $\tilde{\Phi}_p^*(z)$. Then, the input/output relationship is given by

$$x(z) = \left(1\, z^{-1} \ldots z^{-(m-1)}\right) \Phi_p(z) \tilde{\Phi}_p^*(z) x_p(z), \quad (40)$$

where $x_p(z)$ is the vector of polyphase components of the signal (there are $m$ of them) and $^*$ denotes conjugation of coefficients but not of $z$. Note that the polyphase components of the analysis bank are defined in reverse order from those of the synthesis bank. When the filter length is $l = m$, then, each polyphase sequence is of length one. Each polyphase matrix then reduces to $\Phi_p(z) = \Phi_0$, $\tilde{\Phi}_p^*(z) = \tilde{\Phi}_0^*$, that is, both $\Phi_p(z)$ and $\tilde{\Phi}_p^*(z)$ become independent of $z$. It is clear from the above, that to obtain perfect reconstruction, that is, to have a basis expansion, the polyphase matrices must satisfy

$$\Phi_p(z) \tilde{\Phi}_p^*(z) = I. \quad (41)$$

If the filter length is $l = m$, the above implements a finite-dimensional expansion (block transform). For example, if we wanted to implement the $\mathrm{DFT}_m$ using a filter bank, we would use an $m$-channel filter bank with sampling by $m$ and prototype synthesis filters $\varphi_i$ given in (11). Since each prototype filter is of length $m$, each of its polyphase components will be of length one and a constant, leading to a constant polyphase matrix.

If a filter bank implements an ONB, then $\tilde{\Phi}_p(z) = \Phi_p(z^{-1})$, and (31) reduces to

$$\Phi_p(z) \Phi_p^*(z^{-1}) = I. \quad (42)$$

A matrix satisfying the above is called a *paraunitary matrix*, that is, it is unitary on the unit circle.

## EXAMPLE

As a first example, go back to the Haar expansion discussed earlier. Since $m = 2$, $\varphi_0(z) = (1 + z^{-1})/\sqrt{2}$, $\varphi_1(z) = (1 - z^{-1})/\sqrt{2}$, and the polyphase matrix is $\Phi_p^*(z) = \Phi_0^*$ from (37).

As a more involved example, suppose $m = 2$ again and we are given the following set of template filters:

$$\begin{aligned}
G(z) &= z^{-2} + 4z^{-1} + 6 + 4z + z^2, \\
H(z) &= \frac{1}{4} z \left( \frac{1}{4} z^{-1} + 1 + \frac{1}{4} z \right), \\
\tilde{G}(z) &= \frac{1}{4} \left( -\frac{1}{4} z^{-1} + 1 - \frac{1}{4} z \right), \\
\tilde{H}(z) &= z^{-1} (z^{-2} - 4z^{-1} + 6 - 4z + z^2).
\end{aligned}$$

Having the polyphase decomposition for each filter being written as $G(z) = G_0(z^2) + z^{-1} G_1(z^2)$, $H(z) = H_0(z^2) + z^{-1} H_1(z^2)$, $\tilde{G}(z) = \tilde{G}_0(z^2) + z\tilde{G}_1(z^2)$, $\tilde{H}(z) = \tilde{H}_0(z^2) + z\tilde{H}_1(z^2)$, the polyphase matrices are then (they have polyphase components of the above filters as their columns)

$$\begin{aligned}
\Phi_p(z) &= \begin{pmatrix} z^{-1} + 6 + z & \frac{1}{16}(1+z) \\ 4(1+z) & \frac{1}{4} z \end{pmatrix}, \\
\tilde{\Phi}_p(z) &= \begin{pmatrix} \frac{1}{4} & -4(1 + z^{-1}) \\ -\frac{1}{16}(1 + z^{-1}) & 1 + 6z^{-1} + z^{-2} \end{pmatrix}.
\end{aligned}$$

Thus, the filter bank with filters as defined above implements a biorthogonal expansion. The dual bases are

$$\begin{aligned}
\Phi &= \{\varphi_{2i}, \varphi_{2i+1}\}_{i \in \mathbb{Z}} = \{\tau_{2i} g, \tau_{2i} h\}_{i \in \mathbb{Z}}, \\
\tilde{\Phi} &= \{\tilde{\varphi}_{2i}, \tilde{\varphi}_{2i+1}\}_{i \in \mathbb{Z}} = \{\tau_{2i} \tilde{g}, \tau_{2i} \tilde{h}\}_{i \in \mathbb{Z}},
\end{aligned}$$

and they are interchangeable. □

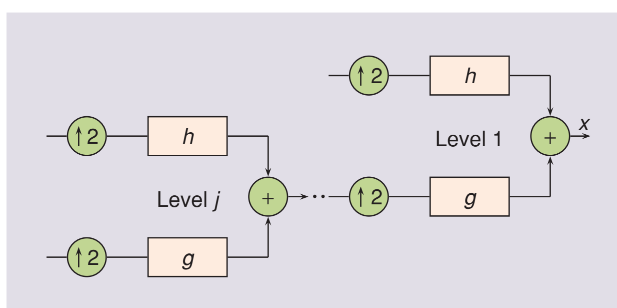
## FILTER BANK TREES

Many of the bases in $\ell^2(\mathbb{Z})$ (and frames later on) are built by using two- and $m$-channel filter banks as building blocks. For example, the dyadic (with scale factor 2) discrete wavelet transform (DWT) is built by iterating the two-channel filter bank on the low-pass channel (Figure 5 depicts the synthesis part). The DWT is a basis expansion and as such nonredundant (critically sampled). To describe the redundancy of various frame families later on, we introduce sampling grids in Figure 5 in Part II of this article [48], each depicting time positions of basis vectors at each level. Thus, for example, the top plot in the same figure depicts the grid for the DWT. At level 1, we have half as many points as at level 0, at level 2, half as many as at level 1, and so on. Because of appropriate sampling, the grid has exactly as many points as needed to represent any $x \in \ell^2(\mathbb{Z})$ and is thus nonredundant.

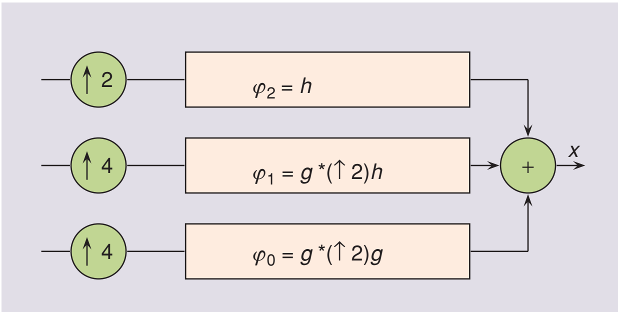
We can also build arbitrary trees by, at each level, iterating on any subset of the branches (typically known as *wavelet packets* [21]). To analyze these tree-structured filter banks, we typically collect all the filters and samplers along a path into a branch with a single filter and single sampler. This is possible using the so-called Noble identities [61] that allow us to exchange the order of filtering and sampling.

## EXAMPLE

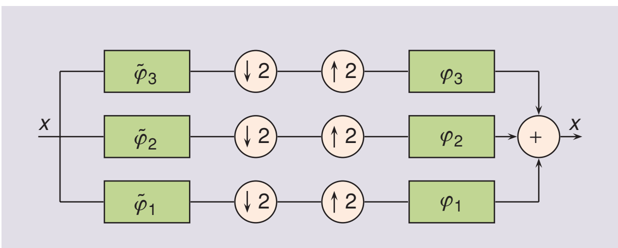
Assume we have a DWT with two levels, that is, the low-pass branch is iterated only once (see Figure 5). Then, the equivalent filter bank has three channels as in Figure 6 with sampling by 2,



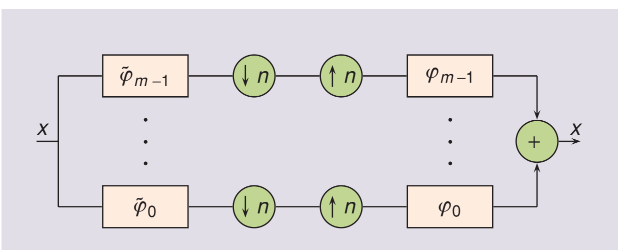

**[FIG5] The synthesis part of the filter bank implementing the DWT with *j* levels. The analysis part is analogous.**




**[FIG6] The synthesis part of the equivalent three-channel filter bank implementing the DWT with two levels. The analysis part is analogous.**




**[FIG7] Three-channel filter bank with downsampling by two.**




**[FIG8] A filter bank implementation of a frame expansion: It is an *m*-channel filter bank with sampling by *n*.**

4, and 4, respectively. The equivalent filters are then (call $(\uparrow m)$ the operator upsampling a filter by $m$)

$$\varphi_2 = h, \quad \varphi_1 = g * (\uparrow 2)h, \quad \varphi_0 = g * (\uparrow 2)g.$$

Assuming for simplicity that the filters have only two taps, the matrix $\Phi$ in (38) is block diagonal with

$$\Phi_0^* = \begin{pmatrix} h_0 & h_1 & 0 & 0 \\ 0 & 0 & h_0 & h_1 \\ g_0 h_0 & g_1 h_0 & g_0 h_1 & g_1 h_1 \\ g_0^2 & g_0 g_1 & g_0 g_1 & g_1^2 \end{pmatrix}.$$

We see here that even though we have only three branches, the filter bank behaves as a critically sampled four-channel filter bank with sampling by four. □

### FILTER BANK VIEW OF FRAMES

The filter bank expansions we just discussed were bases and thus nonredundant. Now, nothing stops us from being redundant (for reasons stated earlier) by simply adding more vectors.

#### EXAMPLE

Let us look at the simplest case using our favorite example: the MB frame given in "The Mercedes-Benz Frame." Our $\Phi^*$ is now block diagonal, with $\Phi_0^* = \Phi_{\mathrm{UNTF}}^*$ from (16) on the diagonal. In contrast to finite-dimensional bases implemented by filter banks [see Haar in (37)], the block $\Phi_0^*$ is now rectangular of size $3 \times 2$. This finite-dimensional frame is equivalent to the filter bank shown in Figure 7 with $\{\tilde{\varphi}_i\} = \{\varphi_i\}$, given in (16). □

As we could for finite-dimensional bases, we can investigate finite-dimensional frames within the filter bank framework (see Figure 8). In other words, all cases we consider in this article, both finite dimensional and infinite dimensional, we can look at as filter banks.

Similarly to bases, if in (38) $\Phi$ is not block diagonal, we resort to the polyphase-domain analysis. Assume that the filter length is $l = kn$ (if not, we can always pad with zeros), and write the frame as (causal filters)

$$\Phi^* = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \Phi_0^* & \Phi_1^* & \cdots & \Phi_{k-1}^* & 0 & \cdots \\ \cdots & 0 & \Phi_0^* & \cdots & \Phi_{k-2}^* & \Phi_{k-1}^* & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & 0 & 0 & \cdots & \Phi_0^* & \Phi_1^* & \cdots \\ \cdots & 0 & 0 & \cdots & 0 & \Phi_0^* & \cdots \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad (43)$$

where each block $\Phi_i$ is of size $n \times m$. $\Phi_0$, for example, is

$$\Phi_0 = \begin{pmatrix} \varphi_{00} & \cdots & \varphi_{0,m-1} \\ \vdots & \ddots & \vdots \\ \varphi_{n-1,0} & \cdots & \varphi_{n-1,m-1} \end{pmatrix}.$$

In the above, we enumerate template frame vectors from $0, \ldots, m-1$. A thorough analysis of oversampled filter banks seen as frames is given in [13], [24], and [25].

### SUMMARY

To summarize, the class of multiresolution transforms obtained using a filter bank depends on three parameters: the number of vectors $m$, the shift or sampling factor $n$, and the length $l$ of the nonzero support of the vectors.

#### BASES $m = n$

The filter bank in this case is called critically sampled and implements a nonredundant expansion—*basis*. The basis $\Phi$ has a dual basis associated with it, $\tilde{\Phi}$, leading to biorthogonal filter banks. The associated matrices $\Phi$, $\tilde{\Phi}$ are invertible. In the $z$-

domain, this is expressed as follows: A filter bank implements a basis expansion if and only if (41) evaluated on the unit circle is satisfied [64].

An important subcase is when the basis $\Phi$ is orthonormal, in which case it is self-dual, that is, $\tilde{\Phi} = \Phi$. The filter bank is called orthogonal and the associated matrix $\Phi$ is unitary, $\Phi\Phi^* = I$. In the $z$-domain, this is expressed as follows: A filter bank implements an ONB expansion if and only if its polyphase matrix is paraunitary, that is, if and only if (42) holds [64]. Well-known subcases are the following:

- When $l = m$, we have a *block transform*. In this case, in (38), only $\Phi_0$ exists, making $\Phi$ block-diagonal. In effect, since there is no overlap between processed signal blocks, this can be analyzed as a finite-dimensional case, where both the input and the output are $m$-dimensional vectors. A famous example is the Discrete Fourier Transform (DFT), which we discussed earlier.
- When $m = 2$, we get two-channel filter banks. In (38), $\Phi_i$ is of size $2 \times 2$ and by iterating on the low-pass channel, we get the DWT [64] (see Figure 5).
- When $l = 2m$, we get *Lapped Orthogonal Transforms* (LOT), efficient transforms developed to deal with the blocking artifacts introduced by block transforms, while keeping the efficient computational algorithm of the DFT [64]. In this case, in (28), only $\Phi_0$ and $\Phi_1$ are nonzero.

### FRAMES $m > n$

The filter bank in this case implements a redundant expansion—*frame*. The frame $\Phi$ has a dual frame associated with it, $\tilde{\Phi}$. The associated matrices $\Phi$, $\tilde{\Phi}$ are rectangular and left/right invertible. This has been formalized in $z$-domain in [24], as the following result: A filter bank implements a frame decomposition in $\ell^2(\mathbb{Z})$ if and only if its polyphase matrix is of full rank on the unit circle.

An important subcase is when the frame $\Phi$ is *tight*, in which case it is self-dual, that is, $\tilde{\Phi} = \Phi$, and the associated matrix $\Phi$ satisfies $\Phi\Phi^* = I$. This has been formalized in $z$-domain in [24], as the following result: A filter bank implements a TF expansion in $\ell^2(\mathbb{Z})$ if and only if its polyphase matrix is paraunitary. A well-known subcase of TFs is the following:

- When $l = n$, we have a block transform. Then, in (43), only $\Phi_0$ is nonzero, making $\Phi$ block diagonal. In effect, since there is no overlap between processed blocks, this can be analyzed as a finite-dimensional case, where both the input and the output are $n$-dimensional vectors.

## CONCLUSIONS

Coming to the end of Part I, we hope you have a different picture of a frame in your mind from a "picture frame." While necessarily colored by our personal bias, we intended this tutorial as a basic introduction to frames, geared primarily toward engineering students and those without extensive mathematical training. Frames are here to stay; as wavelet bases before them, they are becoming a standard tool in the signal processing toolbox, spurred by a host of recent applications requiring some level of redundancy. Part II [48] covers these applications, so stay tuned…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