ECE 8823a, Spring 2011

Homework #3

Due Wed March 16, a the beginning of class

Suggested Reading:

- Mallat, sections 7.1 and 7.2

- For a very in depth discussion on wavelets and wavelet design, see Daubechies' book *Ten Lectures on Wavelets*

Problems:

1. Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Meyer Wavelets. Let the filter $h[n]$ be defined by its discrete-time Fourier transform

$$\hat{h}(\omega) = \begin{cases} \sqrt{2} & |\omega| \leq \pi/3 \\ \sqrt{2}\beta\left(3 - \frac{6|\omega|}{\pi}\right) & \pi/3 \leq |\omega| \leq 2\pi/3 \\ 0 & 2\pi/3 \leq |\omega| \leq \pi \end{cases},$$

where $\beta(u)$ is a window function which obeys $\beta(u) = 0$ for $u < -1$, $\beta(u) = 1$ for $u > 1$, is monotonic between -1 and 1, and $\beta^2(u) + \beta^2(-u) = 1$.

(a) Verify that the filter h satisfies the necessary and sufficient conditions to generate and MRA. These properties are on page 50 of the notes, and are given in Theorem 7.2 (page 271) in Mallat.

(b) Sketch the Fourier transforms of the scaling function $\phi(t)$ and wavelet function $\psi(t)$.

(c) Verify directly that the $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ are orthogonal by showing $\langle\phi(t-k), \phi(t-\ell)\rangle = 0$ unless $k = \ell$.

(d) Verify directly that V_0 is orthogonal to W_0 by showing $\langle\phi(t-k), \psi(t-\ell)\rangle = 0$ for all k, ℓ.

3. Daubechies Wavelets. In class, we saw that designing a compactly supported wavelet $\psi(t)$ with p vanishing moments is equivalent to designing an FIR filter $h[n]$ whose frequency response has the form

$$\hat{h}(\omega) = (1 + e^{-j\omega})^p Q(e^{-j\omega}),$$

where $Q(\cdot)$ is a polynomial chosen carefully such that

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega + \pi)|^2 = 2.$$

(a) Write a MATLAB function of the form

```
R = daubpoly(p)
```

that takes a number of vanishing moments p and returns the coefficients for the polynomial $R(z) = Q(z)Q(z^{-1})$. Recall that we did the special case of $p = 3$ in class (page II.70–71 of the notes). For the general case, you will have to set up the $(2p-1) \times (2p-1)$ system of equations and solve them. The entries of the matrix will contain the binomial coefficients of order $2p$; I have provided the function `binom.m` to compute these. Turn in your code, and pole-zero plots of R for $p = 4, 6, 8$.

(b) Write a MATLAB function of the form

```
[h,g] = daubfilt(p)
```

that takes a number of vanishing moments p and returns the "minimum phase" filter h and its complement $g[n] = (-1)^{1-n}h[1 - n]$. Of course you will use your function `daubpoly` from the previous part. Recall that "minimum phase" means that when we factor $R(z) = Q(z)Q(z^{-1})$, we assign all the roots inside the unit circle to $Q(z)$. Notice that `roots(R)` returns the roots of R sorted by magnitude.

Make sure your filter coefficients are normalized so that $\hat{h}(0) = \sum_n h[n] = \sqrt{2}$. Check that your code is correct by comparing against page 295 of Mallat (or page II.77 of the notes). Turn in

   i. your code,
   ii. print-outs of the filter coefficients $h[n]$ and $g[n]$ to 12 digits (use `format long`) for $p = 4$ and $p = 10$, and
   iii. pole-zero plots for h and g for $p = 4$ and $p = 10$.

(c) Use the function `plot_phi_psi.m` that I have provided to plot the wavelets and scaling functions for $p = 4, 6, 8$. This is easy given the work above; simply call

```
plot_phi_psi(h);
```

and then print out the figures.

(d) *Optional ungraded additional work for the enterprising student:* Modify your code for part (a) so that it solves a $p \times p$ system instead of a $(2p - 1) \times (2p - 1)$ system. This is done simply by taking advantage of the fact that we know $R(z)$ will be symmetric in that if we write

$$R(z) = \sum_k r_k z^k,$$

we will have $r_{-k} = r_k$. (In the end, this really isn't that hard.)

4. Symmlets. There are other ways to factor $R(z)$ into $Q(z)Q(z^{-1})$ other than "minimum phase". For example, consider the pole-zero plot of $G(z) = (1 + z)^{2p}R(z)$ for $p = 7$ shown below:
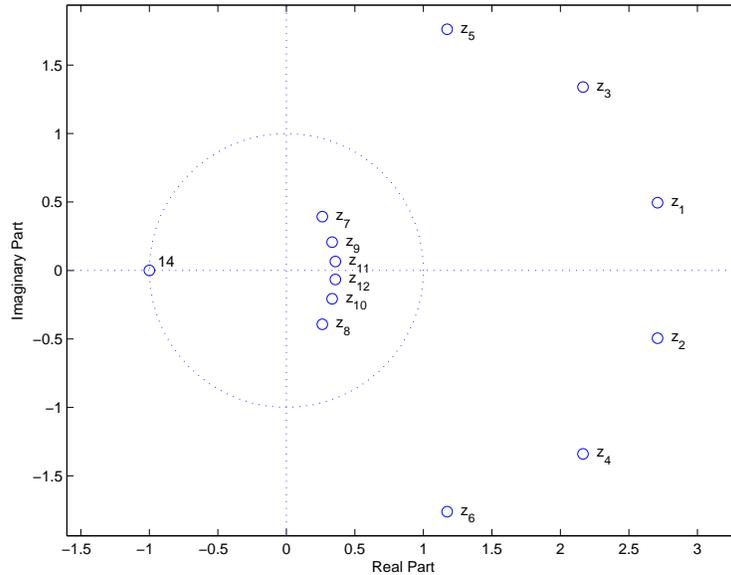
The roots $z_1, \ldots, z_{12}$ of $R(z)$ are labeled in the order in which the MATLAB command `roots(R)` returns them. Of course, if we assign z_k to $Q(z)$, we must also assign z_k^{-1} to $Q(z^{-1})$ (and so in the figure above, if z_7 goes to $Q(z)$, then z_5 must go to $Q(z^{-1})$, and if z_1 goes to $Q(z)$ then z_{11} must go to $Q(z^{01})$, etc.). Also, because we want the coefficients of $Q(z)$ to be real, if we assign z_k to $Q(z)$ we must also assign z_k^* to $Q(z)$ (and so in the figure above, (z_7, z_8) get assigned together, as do (z_1, z_2), etc.). With these constraints, the factorization is completely determined by where we assign z_7, z_9, and z_{11}, and so there are $2^3 = 8$ total possible factorizations.

There are even fewer if we consider $h[n]$ and $h[-n]$ to be the same (as they produce scaling functions and wavelets that have the same essential shape, just reflected). For instance, the factorizations
$$Q(z) = (1 - z_7)(1 - z_8)(1 - z_3)(1 - z_4)(1 - z_1)(1 - z_2)$$
and
$$Q(z) = (1 - z_5)(1 - z_6)(1 - z_9)(1 - z_{10})(1 - z_{11})(1 - z_{12})$$
lead to polynomials (and eventually filters h) that are just time-reversals of one another (try it, if you don't believe me). Thus there are really only 4 unique factorizations.

Find these four unique factorizations, plot the scaling function and wavelet associated with each (put all plots on different axes, but the same page) using `plot_phi_psi.m`. Which one has the wavelet which looks the most symmetric?

5. Mallat Problem 7.8. Let f be a function of support $[0, 1]$ that is equal to different polynomials of degree q on the intervals $\{[\tau_k, \tau_{k+1}]\}_{0 \leq k \leq K}$, with $\tau_0 = 0$ and $\tau_K = 1$. Let ψ be a Daubechies wavelet with p vanishing moments. Depending on p, compute the number of possible non-zero wavelet coefficients $\langle f, \psi_{j,n} \rangle$ for a given scale j. How should we choose p to minimize this number?