ECE 8823a, Spring 2011

Homework #2

Due Wednesday February 23, <u>AT THE BEGINNING</u> of class

Reading: Mallat 8.3 and 8.4

1. Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. Let $\{\phi_k\}_{k=1}^m$ be a frame for $\mathbb{C}^n$ with the standard ℓ_2 inner product, and suppose that $\|\phi_k\|_2^2 = 1$. Show that the frame bounds for $\{\phi_k\}_{k=1}^m$ obey $A \le m/n \le B$.

3. Let $f(t)$ be a continuous-time signal. Prove that $\hat{f} \in L_1$ implies $f(t)$ is uniformly continuous, where $\hat{f}(\omega)$ is the Fourier transform of $f(t)$. That is, given an $\epsilon > 0$, find a $\delta > 0$ independent of t such that
$$|f(t) - f(t+h)| < \epsilon \quad \text{for all} \quad |h| \le \delta.$$
You might find it helpful to define the function
$$I(\Omega) = \int_{|\omega|>\Omega} |\hat{f}(\omega)| \, d\omega.$$
Notice that $I(0) = \|\hat{f}\|_{L_1}$ and that $I(\Omega) \to 0$ as $\Omega \to \infty$. Since we are trying to show that $f(t)$ is *uniformly* continuous, your choice of δ should not depend on t. It will, however, probably depend on $I(\Omega)$ (and of course ϵ).

4. Let $\mathcal{E}$ be the space of finite energy functions that are even, $f(-t) = f(t)$, and let $\mathcal{O}$ be the space of finite energy functions that are odd, $f(-t) = -f(t)$.
   (a) Given a fixed $f(t) \in L_2(\mathbb{R})$, find $f_e(t) \in \mathcal{E}$ that minimizes
   $$\min_{g \in \mathcal{E}} \|f - g\|_{L_2}^2,$$
   and $f_o(t) \in \mathcal{O}$ that minimizes
   $$\min_{g \in \mathcal{O}} \|f - g\|_{L_2}^2.$$
   Defend your answers vigorously.
   (b) Show that $f_e(t) + f_o(t) = f(t)$. Conclude that your answers to part (a) define orthogonal projectors onto a partition of $L_2(\mathbb{R})$.
   (c) Let $\{\phi_k\}_{k\in\Gamma}$ be an orthobasis for $L_2([0,1])$. Show how to use $\{\phi_k\}_{k\in\Gamma}$ to construct an orthobasis for $\mathcal{E}([-1,1])$ (the space of even functions on the interval $[-1,1]$) and an orthobasis for $\mathcal{O}([-1,1])$.

5. Let P^+ and P^- be the operators define on page II.17 of the notes. Show that they are orthoprojectors onto the spaces W^+ and W^-, respectively. That is, show that $\mathrm{P}^+ f$ solves the optimization program

$$\min_{g \in W^+} \|f - g\|_{L_2}^2,$$

and similarly for $P^- f$. Do this by verifying the orthogonality condition. This will become very easy if you establish that P^+ is self-adjoint and $\mathrm{P}^+ \mathrm{P}^+ = \mathrm{P}^+$ (and similarly for P^-).

6. The file `beta.mat` contains a window function, indexed by $n = 1, 2, \ldots, 256$, which satisfies

$$\beta(n) = \begin{cases} 0 & 1 \leq n \leq 64, \\ 1 & 193 \leq n \leq 256, \end{cases}$$

is monotonically increasing on $65 \leq n \leq 192$, and

$$\beta^2(n) + \beta^2(256 - n + 1) = 1, \quad \forall\, 1 \leq n \leq 256.$$

(a) Create a lapped orthobasis for $\mathbb{R}^{256}$ with 128 basis vectors of the form

$$\phi_k(n) = \beta(n) \cdot (\text{cosine function}), \quad k = 1, \ldots, 128$$

and 128 basis vectors of the form

$$\phi_k(n) = \beta(256 - n + 1) \cdot (\text{cosine function}), \quad k = 129, \ldots, 256.$$

Demonstrate in MATLAB that the basis you have constructed in indeed orthonormal (by verifying that $\Phi^*\Phi = I$), and plot a few representative $\phi_k(n)$.

(b) Create the signal $f(n)$ as follows:

```
t = linspace(-3,3,256);  f(t) = exp(-t.^ 2);
```

Compare the expansion coefficients $\alpha = \Phi f$ using the orthobasis you constructed above to the "short-time DFT coefficients" created by taking length-128 FFTs of `f(1:128)` and `f(129:256)` and concatenating the coefficients. On the same plot, show the sorted (in decreasing order) magnitudes of the expansion in each basis. Comment on the ability of each to create accurate approximations of $f(n)$.