ECE 8823a, Spring 2011

Homework #1

Due Wednesday February 9, at the beginning of class

Suggested reading:

- Young, Chapters 1–4. These four chapters contain a beautiful exposition of the major results pertaining to orthogonal decompositions in Hilbert spaces.
- Chapter 5 in Young shows that Fourier series is complete in $L_2([0,T])$, take a look at how that is established if you are interested. Chapter 7 contains the basics of linear operators between Hilbert spaces — in this class, we will only use a small portion of this material, but it is great stuff. Chapter 6 discusses dual spaces; we will not talk about these at all, but again these ideas are of broad interest.
- The paper handed out in class, "Life beyond bases: The Advent of Frames, Part I" by Kovecevic and Chebira gives a very nice introduction to frame theory from a signal processing perspective. Part II (which is available on the web page) talks about some applications where frames accomplish things that orthobases cannot.
- Mallat, Chapter 5 contains an introduction to basis/frame decompositions that is very attuned to the level of this course.
- Heil's "A Basis Theory Primer" gives a much more rigorous treatment of the material we have encountered so far (Christopher Heil is a professor in the mathematics department here at Georgia Tech). This has just been published as a book; an older version of the manuscript can be downloaded off the course web page.
- The 1986 paper "Painless nonorthogonal expansions"by Daubechies, Grossman, and Meyer (available from the course web site) was one of the first papers to talk about overcomplete frame expansions in the manner in which we have become accustomed.

Problems:

1. Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

2. (a) Suppose that D is an $n \times n$ diagonal matrix:

$$D = \mathrm{diag}(\{d_1, d_2, \ldots, d_n\}), \quad d_k \in \mathbb{C}.$$

Show that[1]

$$\min_k |d_k|^2 \;\le\; \|Dx\|_2^2 \;\le\; \max_k |d_k|^2 \quad \text{for} \quad \|x\|_2 = 1.$$

Describe the $x \in \mathbb{C}^n$ that obtain these extremal values.

[1] We will sometimes use $\|\cdot\|_p$ for the ℓ_p or L_p norms.

(b) Let A be a $m \times n$ matrix with $m \geq n$. Show that

$$\sigma_n(A)^2 \;\leq\; \|Ax\|_2^2 \;\leq\; \sigma_1(A)^2 \quad \text{for} \quad \|x\|_2 = 1,$$

where $\sigma_i(A)$ is the ith largest singular value of the matrix A.

(c) Suppose that D is a (linear) operator that maps $L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by pointwise multiplication. That is,

$$D[f(t)] = d(t)f(t),$$

for some complex-valued function $d(t)$. Show that

$$\inf_t |d(t)|^2 \;\leq\; \|D[f]\|_2^2 \;\leq\; \sup_t |d(t)|^2 \quad \text{for} \quad \|f\|_2 = 1.$$

3. Let $H : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ be a convolution operator with "kernel" $\{h_k\}_{k\in\mathbb{Z}}$:

$$(H[x])_k = \sum_{m=-\infty}^{\infty} h_{k-m}x_m.$$

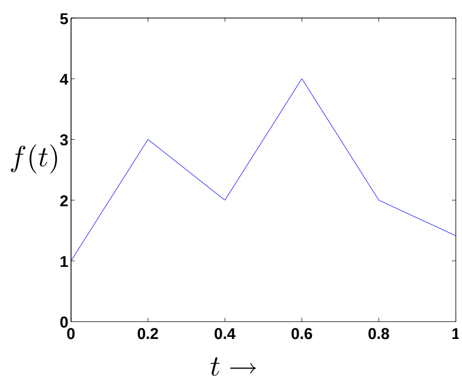
Show that

$$\left(\inf_{\omega\in[-\pi,\pi]} |\hat{h}(\omega)|^2\right) \|x\|_2^2 \;\leq\; \|H[x]\|_2^2 \;\leq\; \left(\sup_{\omega\in[-\pi,\pi]} |\hat{h}(\omega)|^2\right) \|x\|_2^2,$$

where $\hat{h}(\omega)$ is the Discrete-time Fourier Transform of h:

$$\hat{h}(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}.$$

4. Let PL_n be the space of functions on $[0, 1]$ that are continuous and piecewise linear with knots at k/n, $k = 0, 1, \ldots, n$. An example of an $f \in \mathrm{PL}_5$ is shown below.



It should be clear that PL_n has dimension $n+1$, and that

$$\phi_0(t) = \begin{cases} \sqrt{3n}\,(1-tn) & 0 \le t \le 1/n \\ 0 & \text{otherwise} \end{cases},$$

$$\phi_k(t) = \begin{cases} \sqrt{3n/2}\,(nt-k+1) & (k-1)/n \le t \le k/n \\ \sqrt{3n/2}(k+1-nt) & k/n \le t \le (k+1)/n \\ 0 & \text{otherwise} \end{cases}, \quad k = 1,\dots,n-1,$$

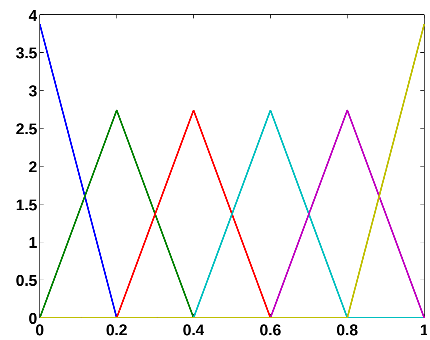
$$\phi_n(t) = \begin{cases} \sqrt{3n}(nt-n+1) & (n-1)/n \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is a basis for PL_n. For example, the signal in PL_5 above can be written as

$$f(t) = \frac{1}{\sqrt{15}} \cdot \phi_0(t) + \frac{3\sqrt{2}}{\sqrt{15}} \cdot \phi_1(t) + \frac{2\sqrt{2}}{\sqrt{15}}\phi_2(t) + \frac{4\sqrt{2}}{\sqrt{15}}\phi_3(t) + \frac{2\sqrt{2}}{\sqrt{15}}\phi_4(t) + \frac{1}{\sqrt{15}}\phi_5(t).$$

where the six basis functions are sketched on the same set of axes below:

$\phi_0(t), \dots, \phi_5(t)$



$t \longrightarrow$

In general we can write $f \in \mathrm{PL}_n$ as $f(t) = \sum_{k=0}^{n} \alpha_k \phi_k(t)$ where the α_k are just scaled samples of $f(t)$ taken at the knots $t = k/n$.

(a) Let $\Phi : \mathrm{PL}_n \to \mathbb{R}^{n+1}$ be the linear operator that maps such a continuous piecewise linear function to its (standard Euclidean) inner products against the ϕ_k:

$$(\Phi[f])_k = \langle f, \phi_k \rangle, \quad k = 0, \dots, n.$$

The adjoint $\Phi^* : \mathbb{R}^{n+1} \to \mathrm{PL}_n$ synthesizes a signal from a set of coefficients $\alpha \in \mathbb{R}^{n+1}$ using

$$(\Phi^*[\alpha])(t) = \sum_{k=0}^{n} \alpha_k \phi_k(t).$$

Then the operator $\Phi\Phi^* : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ can be represented by an $n+1 \times n+1$ matrix. Write an expression for each entry $(\Phi\Phi^*)_{k,m},\ 0 \le k, m \le n$ in this matrix.

(b) Write a Matlab function that computes the matrix $\Phi\Phi^*$ for a fixed n. Use this function along with the `eig` command to calculate the frame bounds for this basis. That is, find the largest A and smallest B such that

$$A\|f\|_{L_2}^2 \;\leq\; \|\Phi[f]\|_{\ell_2}^2 = \sum_{k=0}^{n} |\langle f, \phi_k\rangle|^2 \;\leq\; B\|f\|_{L_2}^2.$$

for $n = 5, 10$, and 100. (Recall that $\|\Phi\| = \|\Phi^*\|$, where $\|\cdot\|$ is the operator norm.) Report the three answers above, and turn in a printout of your code. Also turn in a plot the spectrum (eigenvalues) of $\Phi\Phi^*$ for $n = 100$.

(c) Write a Matlab script that calculates and plots the dual basis vectors $\{\tilde{\phi}_k\}$ for $n = 5$. The operator Φ is not a matrix, so you cannot just invert it. But you could easily have Matlab plot any linear combination of the ϕ_k. So you have to figure out how to find $\{\alpha_{k,j}$ so that $\tilde{\phi}_k(t) = \sum_j \alpha_{k,j}\phi_j(t)$. Turn in a print out of your script and the plots (on six separate axes, but all on one page, please).

5. We now consider an infinite-dimensional version of the previous problem. Let $\mathrm{PL}(\mathbb{R})$ be the set of finite-energy, continuous, and piecewise-linear functions on the real line with knots at the integers. That is, $f \in \mathrm{PL}(\mathbb{R})$ have $\|f\|_{L_2} < \infty$ and obey

$$f(t) = (k-t)f(k-1) + (t-k+1)f(k), \quad \text{for} \quad k-1 \leq t \leq k.$$

It should be clear that $\{\phi_k\}_{k\in\mathbb{Z}}$, where

$$\phi_k(t) = \begin{cases} \sqrt{3/2}(t-k+1) & k-1 \leq t \leq k \\ \sqrt{3/2}(k+1-t) & k \leq t \leq k+1 \\ 0 & \text{otherwise} \end{cases},$$

is a basis for $\mathrm{PL}(\mathbb{R})$. Calculate (by hand) the frame bounds. That is, find the largest A and smallest B such that

$$A\|f\|_{L_2}^2 \;\leq\; \sum_{k=-\infty}^{\infty} |\langle f, \phi_k\rangle|^2 \;\leq\; B\|f\|_{L_2}^2, \quad \text{for all} \quad f \in \mathrm{PL}(\mathbb{R}).$$

(Hint: Problem 3.)

6. It is often the case that the *smoothness* of a signal is reflected in how quickly its transform coefficients decay. In this problem, we will introduce a simple notion of smoothness, called *Lipchitz continuity*, and show that if a signal has this property, then its Haar coefficients get small very quickly as the scale j increases.

A function $f(t)$ on $[0,1]$ is called *Lipschitz* if there exists a $K > 0$ such that

$$|f(t) - f(v)| \;\leq\; K|t-v| \quad \text{for all } t, v, \in [0,1].$$

The K above is called the *Lipschitz constant*. Another way to write this is

$$|f(t) - f(t+h)| \;\leq\; Kh \quad \text{for all } t, t+h \in [0,1].$$

Lipschitz continuity is stronger than general continuity; we are not only asking that the left hand side above go to zero as $h \to 0$, we are asking that it go to zero faster than Kh. Lipschitz functions are essentially differentiable; their derivatives exist and are bounded everywhere except at a small number of points (a set of "measure zero", if you know something about measure theory). The Wikipedia article on Lipschitz Continuity makes good reading if you want to know more.

(a) Define the Haar orthobasis $\{\phi_{j,k}\}$ for $L_2([0,1])$ as defined on page I.18 of the notes. Let f be Lipschitz with constant K. Show that there is a another constant C such that

$$|\langle f, \phi_{j,k}\rangle| \ \leq \ C \cdot 2^{-3j/2}.$$

Compute C explicitly. It should depend on K, but not the indices j or k. Start off by writing

$$\langle f, \phi_{j,k}\rangle = \text{Const}\left(\int_{t_1}^{t_2} f(t)\, dt - \int_{t_2}^{t_3} f(t)\, dt\right) = \text{Const}\left(\int_{t_1}^{t_2} f(t) - f(t+t_2-t_1)\, dt\right),$$

for some appropriate constants and t_i, and then apply the definition of Lipschitz.

(b) Find a bound on the ℓ_2 norm of the Haar coefficients at scale j:

$$E(j) = \sum_{k=0}^{2^{j-1}-1} |\langle f, \phi_{j,k}\rangle|^2.$$

(c) Let $\tilde{f}_J$ be the projection of f on to the space

$$V_J = \text{span}(\{\phi_{j,k},\ j = 0, \ldots, J,\ k = 0, \ldots, 2^{j-1}-1\}).$$

Of course, we can write

$$\tilde{f}_J(t) = \sum_{j=0}^{J} \sum_{k=0}^{2^{j-1}-1} \langle f(t), \phi_{j,k}(t)\rangle\ \phi_{j,k}(t).$$

Using your results from (a) and (b), find a bound on the approximation error

$$A(J) = \|f - \tilde{f}_J\|_{L_2}.$$

How fast does $A(J) \to 0$ as $J \to \infty$?

7. This problem explores sampling a bandlimited trigonometic polynomial. Suppose $f \in L_2([0,1])$ is both periodic and bandlimited; that is, we can represent f with just $2M+1$ Fourier coefficients:

$$f(t) = \sum_{k=-M}^{M} c_k e^{j2\pi kt}. \quad (1)$$

$f(t)$ is finite dimensional; it is specified in full by the $n = 2M+1$ values of c_k. A modified version of the Shannon sampling theorem tells that we can also reconstruct $f(t)$ from n equally spaced samples between $t = 0$ and $t = 1$, and in fact these samples can be viewed as the expansion coefficients in an orthobasis.

(a) Write a Matlab routine which constructs a $m \times n$ matrix S which takes the $\{c_k\}$ and returns samples of $f(t)$ at m *arbitrary* locations between 0 and 1. Normalize the rows of S so that when $m = n$ and the samples are equispaced, all its singular values are 1 (that is, all of the eigenvalues of S^*S are equal to 1). The function should take as input a set of sample locations $t_1, \ldots, t_m$, and the value of n.

(b) The file `samplelocs.mat` contains a variable `c` of length 7 containing the Fourier series coefficients for a signal f, and a varible T also of length 7 which contains a series of sample locations. Use you code developed in part (a) to return the samples of f at the locations in T.

(c) Let B_M denote the space of bandlimited signals which can be represented as in (1) above, and suppose that the sampling locations $t_1, \ldots, t_m$ are fixed. We will use $\Phi : B_M \to \mathbb{C}^m$ to denote the operator which takes a bandlimited function and returns the m samples at the locations t_k, and $S : \mathbb{C}^n \to \mathbb{C}^m$ to denote the $m \times n$ matrix that takes the Fourier coefficients $\{c_k\}$ to the m samples. Argue that the frame bounds for Φ are the same as those for S.

(d) For the sampling locations in T, calculate and plot (using Matlab and `subplot`) the seven "rows" $\phi_1, \ldots, \phi_7$ of Φ (these are actually functions); we will have $f(t_k) = \langle f, \phi_k \rangle$.

(e) Calculate the frame bounds A and B for Φ.

(f) Calculate and plot the seven dual functions $\tilde{\phi}_1, \ldots, \tilde{\phi}_7$.

(g) Take a moderate value of M, say $M = 100, n = 201$, and take $m = n$. Start with a uniform sampling $\{t_k\}$, and then move two of the samples closer and closer together. Discuss what happens to the frame bounds for Φ as the samples get close, and the implications of this on the recovery of $f(t)$.

(h) Suppose our sample locations are *jittered*; we use

$$t_k = k/m + \Delta_k,$$

where the Δ_k are independent and uniformly distributed:

$$\Delta_k \sim \text{Uniform}[-\tau, \tau]$$

for some given τ. Using Matlab simulations, discuss how the average behavior of the frame bounds A, B and the condition number B/A change as you vary $\tau > 0$ and m (keep n fixed).