Painless nonorthogonal expansions[a)]

Ingrid Daubechies[b)]
Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B 1050 Brussels, Belgium

A. Grossmann
Centre de Physique Theorique II, CNRS-Luminy, Case 907, F 13288 Marseille Cedex 9, France

Y. Meyer
Centre de Mathématiques, Ecole Polytechnique, 91128 Palaiseau Cedex, France

(Received 18 September 1985; accepted for publication 11 December 1985)

In a Hilbert space $\mathscr{H}$, discrete families of vectors $\{h_j\}$ with the property that $f = \Sigma_j \langle h_j | f \rangle h_j$ for every f in $\mathscr{H}$ are considered. This expansion formula is obviously true if the family is an orthonormal basis of $\mathscr{H}$, but also can hold in situations where the h_j are not mutually orthogonal and are "overcomplete." The two classes of examples studied here are (i) appropriate sets of Weyl–Heisenberg coherent states, based on certain (non-Gaussian) fiducial vectors, and (ii) analogous families of affine coherent states. It is believed, that such "quasiorthogonal expansions" will be a useful tool in many areas of theoretical physics and applied mathematics.

I. INTRODUCTION

A classical procedure of applied mathematics is to store some incoming information, given by a function $f(x)$ (where x is a continuous variable, which may be, e.g., the time) as a discrete table of numbers $\langle g_j | f \rangle = \int dx\, g_j(x) f(x)$ rather than in its original (sampled) form. In order to have a mathematical framework for all this, we shall assume that the possible functions f are elements of a Hilbert space $\mathscr{H}$ [we take here $\mathscr{H} = L^2(\mathbf{R})$]; the functions g_j are also assumed to be elements of this Hilbert space.

One can, of course, choose the functions g_j so that the family $\{g_j\}$ ($j \in J$, J a denumerable set) is an orthonormal basis of $\mathscr{H}$. The decomposition of f into the g_j is then quite straightforward: one has

$$f = \sum_j \langle g_j | f \rangle g_j ,$$

where the series converges strongly. The requirement that the g_j be orthonormal leads, however, to some less desirable features. Let us illustrate these by means of two examples.

Take first $g_j(x) = p_j(x)\, w(x)^{1/2}$, where the p_j are orthonormal polynomials with respect to the weight function w. In this case local changes of the function f will affect the whole table of numbers $\langle g_j | f \rangle$ ($j \in J$), which is a feature we would like to avoid.

An orthonormal basis $\{g_j\}$ ($j \in J$), which would enable us to keep nonlocality under control, is given by our second example. We cut $\mathbf{R}$ (the set of real numbers, which is the range of the continuous variable x) into disjoint intervals of equal length, and we construct the g_j starting from an orthonormal basis for one interval. Schematically, consider h_n, an orthonormal basis of $L^2([0,a))$,

$$J = \{(n,m);\ n,m \in \mathbf{Z}, \text{ the set of integers}),$$

$$g_{n,m}(x) = \begin{cases} h_n(x - ma), & \text{for } ma < x < (m+1)a, \\ 0, & \text{otherwise.} \end{cases}$$

If now the function f undergoes a local change, confined to the interval $[ka, la]$, only the numbers $\langle g_{n,m} | f \rangle$ with $k \leq m \leq l - 1$ will be affected, reflecting the locality of the change. This choice for the g_j also has, however, its drawbacks: some of the functions g_j are likely to be discontinuous at the edges of the intervals, thereby introducing discontinuities in the analysis of f, which need not have been present in f itself. This is particularly noticeable if one takes the following natural choice for the h_n:

$$h_n(x) = a^{-1/2} e^{i2\pi x/a} .$$

In this case even very smooth functions f will give values $g_{n,m}(f)$ significantly different from zero for rather high values of n, reflecting high-frequency components artificially introduced by the cutting of $\mathbf{R}$ into intervals.

We shall now see how these undesirable features can be avoided by taking radically different options for the choice of the g_j. In particular, we shall not restrict ourselves to orthonormal bases. Let us start by asking which properties we want to require for the g_j.

The storage of the function f in the form of a discrete table of numbers $\langle g_j | f \rangle$ ($j \in J$) only makes sense if one is certain that f is completely characterized by the numbers $\langle g_j | f \rangle$ ($j \in J$). In other words, we want

$$\langle g_j | f \rangle = \langle g_j | h \rangle, \quad \text{for all } j \text{ in } J,$$

to imply $f = h$, which is equivalent to saying that the vectors $\{g_j\}$ ($j \in J$) span a dense set, i.e., that the orthonormal complement $\{g_j; j \in J\} = \{0\}$. This will be our first requirement.

In all the cases we shall discuss, the set $\{g_j\}$ ($j \in J$) is such that the map

$$T: \ f \to (\langle g_j | f \rangle)_{j \in J}$$

defines a bounded operator from $\mathscr{H}$ to $l^2(J)$, the Hilbert space of all square integrable sequences labeled by J. In other

[a)] This work was carried out within the framework of the R. C. P. "Ondelettes."

[b)] "Bevoegdverklaard Navorser" at the National Foundation for Scientific Research, Belgium.

 0022-2488/86/051271-13$02.50

words, J is countable, and there exists a positive number B such that for all f in $\mathscr{H}$ one has

$$\sum_{j\in J}|\langle g_j|f\rangle|^2<B\,\|f\|^2\,.$$

This can also be stated in the following, equivalent, form: If $|g_j\rangle\,\langle g_j|$ is defined as the operator associating to every vector h in $\mathscr{H}$ the vector $\langle g_j|h\rangle g_j$, then

$$\sum_{j\in J}|g_j\rangle\,\langle g_j|\in\mathscr{B}(\mathscr{H})$$

(the set of all bounded operators in $\mathscr{H}$), with

$$\|\sum_{j\in J}|g_j\rangle\,\langle g_j|\|<B\,.$$

In order to reconstruct f from the discrete table $(\langle g_j|f\rangle)_{j\in J}$, one needs to invert the map T

$$T:\quad f\to(\langle g_j|f\rangle)_{j\in J}$$

from $\mathscr{H}$ to $l^2=l^2(J)$.

In general the image $T\mathscr{H}$ is not all of l^2, but only a subspace of l^2; one can see this, for instance, if the g_j constitute what is often called, in the physics literature, an "overcomplete" set, i.e., if each g_j is in the closed linear span of the remaining ones: $\{g_k;\ k\in J,\ k\neq j\}$. Strictly speaking, there is then no inverse map T^{-1}. This is, of course, no real difficulty: One can define a map $\tilde{T}$ from $l^2(J)$ to $\mathscr{H}$, which is zero on the orthogonal complement of $T\mathscr{H}$ and which inverts T when restricted to $T\mathscr{H}$.

If the spectrum of the positive operator $\Sigma_{j\in J}|g_j\rangle\,\langle g_j|$ reaches down to zero, this inverse is an unbounded operator, and the recovery of f from $(\langle g_j|f\rangle)_{j\in J}$ becomes an ill-posed problem. This is avoided if we require the spectrum of $\Sigma_{j\in J}|g_j\rangle\,\langle g_j|$ to be bounded away from zero, i.e., if we impose that there exist positive constants A,B such that

$$A\mathbf{1}<\sum_{j\in J}|g_j\rangle\,\langle g_j|<B\mathbf{1}\,.$$

(Here $\mathbf{1}$ is the identity operator in $\mathscr{H}$. The inequality sign $<$ between two operators L and T means that their difference $T-L$ is positive definite.) Equivalently, for all f in $\mathscr{H}$, we require

$$A\,\|f\|^2<\sum_{j\in J}|\langle g_j|f\rangle|^2<B\,\|f\|^2\,. \tag{1.1}$$

This is the second condition we impose on the set $\{g_j\}$ $(j\in J)$. The only new condition is the lower bound.

A set of vectors $\{g_j\}$ $(j\in J)$ in a Hilbert space $\mathscr{H}$, satisfying condition (1.1) with $A,B>0$, is called a *frame*.[1] Note that in general the vectors $\{g_j\}_{j\in J}$ will not be a basis in the technical sense, even though their closed linear span is all of $\mathscr{H}$. This is so because the vectors g_j need not be "ω independent," even though they will usually be linearly independent. That is, a vector g_j usually cannot be written as a finite linear combination of vectors $g_{j'}$ (with $j'\neq j$) but it may well belong to the closed linear span of the infinitely many remaining members of the family. Frames were introduced in the context of nonharmonic Fourier series, where the functions g_j are exponentials.[1,2] As far as we know, this is the only context in which frames have been put to use. One of the aims of the present paper is to provide examples of frames in other contexts. Notice that the results on frames in connection with nonharmonic Fourier series can be rewritten as estimates for entire functions in the Paley–Wiener space[1,2]; one of the results we shall derive here can be rewritten as an analogous estimate for entire functions of growth less than $(2,\frac{1}{2})$ (see Ref. 3).

Notice that, even for functions g_j satisfying the condition (1.1), the effective inversion of the map T: $f\to(\langle g_j|f\rangle)_j$ may be a complicated matter. The condition (1.1) on the g_j ensures that the operator is $\tilde{T}$ is bounded $(\|\tilde{T}\|<A^{-1/2})$ but does not provide a way of calculating it. We are still left with a problem where we have to invert large matrices, although some convergence questions are under control. Assuming for a moment that $\tilde{T}$ is given, we may define the family $e_k=\tilde{T}d_k$, where the d_k $(k\in J)$ form the natural orthonormal basis of $l^2(J)$. For $c=(c_j)_{j\in J}\in l^2$, the image $\tilde{T}c$ is then given by

$$\tilde{T}c=\sum_j c_j e_j\,,$$

where the series converges strongly, by the boundedness of $\tilde{T}$. This then implies, for all f in $\mathscr{H}$,

$$f=\sum_j\langle g_j|f\rangle e_j\,, \tag{1.2}$$

again with strong convergence of the series. While (1.2) looks identical to the familiar expansion of f into biorthogonal bases, it really is very different because the $(g_j)_{j\in J}$ need not be a basis at all, technically speaking.

There exists, however, a particular class of frames for which these computational problems do not arise. These are the frames for which the ratio B/A reaches its "optimal" value, $B/A=1$. One has then, for all f in $\mathscr{H}$,

$$\sum_{j\in J}|\langle g_j|f\rangle|^2=A\,\|f\|^2 \tag{1.3}$$

or, equivalently,

$$\sum_{j\in J}|g_j\rangle\,\langle g_j|=A\mathbf{1}\,.$$

So the map T is now a multiple of an isometry from $\mathscr{H}$ into l^2; as such, it is inverted, on its range, by a multiple of its adjoint T^*. Moreover TT^* is a multiple of the orthogonal projection operator on the range T, which can be thus easily characterized.

It is evident that (1.3) is satisfied whenever the g_j constitute an g_j constitute an orthonormal basis (with $A=1$ then). We shall see that there are other, more interesting examples of frames satisfying (1.3), in which the vectors g_j are not mutually orthogonal, and where the set $\{g_j\}$ $(j\in J)$ is "overcomplete" in the sense defined above. We shall say that a frame is *tight* if it satisfies condition (1.3) or, equivalently, if the inequalities in (1.1) can be tightened into equalities. The inversion formula allowing one to recover the vector f from $(\langle g_j|f\rangle)_{j\in J}$ is particularly simple for tight frames. For any f in $\mathscr{H}$ one has

$$f=A^{-1}\sum_{j\in J}\langle g_j|f\rangle g_j\,, \tag{1.4}$$

where the series converges strongly (as in the case of a general frame). The expansion (1.4) is thus entirely analogous

to an expansion with respect to an orthonormal basis, even though the g_j need not be orthogonal. We believe that tight frames and the associated simple (painless!) quasiorthogonal expansions will turn out to be very useful in various questions of signal analysis, and in other domains of applied mathematics. Closely related expansions have already been used in the analysis of seismic signals.[4]

The vectors g_j constituting a tight frame need not be normalized. On the other hand, an orthogonal basis consisting of vectors of different norm, does not constitute a tight frame.

In real life, of course, one will have to deal with finite sets of vectors g_j, i.e., one will have to truncate the infinite set J to a finite subset. The reconstruction problem then becomes ill-posed, and extra conditions, using *a priori* information on f, will be needed to stabilize the reconstruction procedure.[5] We shall not address this question here.

In this paper, we shall discuss two classes of examples of sets $\{g_j\}$ ($j \in J$). In both cases, this discrete set of vectors is obtained as a discrete subset of a continuous family which forms an orbit of a unitary representation of a particular group. Schematically, such families can be described as follows. Consider the following.[6]

(i) $U(\cdot)$ is an irreducible unitary representation, on $\mathscr{H}$, of a locally compact group $\mathscr{G}$.

(ii) $d\mu(\cdot)$ is the left-invariant measure on $\mathscr{G}$.

(iii) Let g be an admissible vector in $\mathscr{H}$ for U (see Ref. 6), i.e., a nonzero vector such that

$$c_g = \|g\|^{-2} \int d\mu(y) |\langle g, U(y) g\rangle|^2 < \infty , \tag{1.5}$$

the integral being taken over $\mathscr{G}$.

(Notice that there are many irreducible unitary representations for which no admissible vectors exist. However, if there is one admissible vector, there is a dense set of them, and we call the representation square integrable.[6])

(iv) Then

$$\int d\mu(y) U(y) |g\rangle \langle g| U(y)^* = c_g \mathbf{1} , \tag{1.6}$$

where the integral is to be understood in the weak sense. If the group $\mathscr{G}$ is unimodular [i.e., if $d\mu(\cdot)$ is both left and right invariant], the existence of one admissible vector in $\mathscr{H}$ implies that all vectors in $\mathscr{H}$ are admissible; moreover, one has in this case that $c_g = c\|g\|^2$ for some c independent of g [see Ref. 6(b)].

(v) In order to obtain possible sets $\{g_j\}$ ($j \in J$) we choose (1) an admissible vector g in $\mathscr{H}$ and (2) a "lattice" of discrete values for the group element y: $\{y_j;\ j \in J\}$.

The vectors g_j are then defined as

$$g_j = U(y_j) g .$$

By imposing appropriate restrictions on g and on J, we shall obtain families $\{g_j\}$ that are frames—or tight frames—in $\mathscr{H}$.

With this procedure it is possible to adjust the "spacing" of the "lattice" $\{y_j;\ j \in J\}$ according to the desired degree of "oversampling." In the two cases that we shall consider, this flexibility can be exploited at little computational cost, since the action of $U(y)$ on g is very simple and the new g_j—obtained after an adjustment of the "lattice"—can be easily and quickly calculated.

In this paper, we shall discuss sets $\{g_j\}$ ($j \in J$) constructed along the lines described above for two different groups; the Weyl–Heisenberg group, and the affine or $ax + b$ group.

In Sec. II we treat the Weyl–Heisenberg case. We start, in Sec. II A, by giving a short review of the definition and main properties of this group and of the associated "overcomplete" set, generally called the set of coherent states. A particular discrete set of coherent states is associated to the so-called von Neumann lattice and to a particular choice of g; it has been discussed and used many times (see, e.g., Refs. 7 and 8). It is well known that the set of coherent states associated to the von Neumann lattice is complete, i.e., that its linear span is dense in $\mathscr{H}$ (see Refs. 8–10). It thus meets the first of the two requirements listed above. We show in Sec. II B that the second requirement is not met: the coherent states associated to the von Neumann lattice do *not* constitute a frame. In Sec. II C we shall see that a similar lattice, with density twice as high, does lead to a frame. In II D we concentrate on analogous families of states based on function g with compact support, as opposed to the most commonly discussed canonical coherent states, where g is a Gaussian. We derive sufficient conditions ensuring that the $g_j = U(y_j) g$ constitute a frame. In Sec. II E we show how g can be chosen in such a way that the frame generated is tight. In Sec. II F we analyze this situation and describe in more detail the necessary and sufficient conditions that g has to satisfy in order to generate a tight frame.

In Sec. III we discuss the $ax + b$ group. Again we start, in Sec. III A, with a short review of definitions and properties, including the so-called affine coherent states. The affine coherent states were first defined in Ref. 11; detailed studies of them can be found, e.g., in Refs. 4 and 12; for applications of these states to signal analysis, see Ref. 4. In Sec. III B we discuss discrete "lattices" of affine coherent states based on "band-limited" functions g, i.e., on functions such that the Fourier transform of g has compact support. We derive sufficient conditions for these discrete sets to be frames. In Sec. III C we show how certain specific choices of g lead to tight frames; in Sec. III D we again analyze the construction, and derive necessary and sufficient conditions on g, ensuring that certain frames will be tight.

As can be readily seen from Sec. II E and Sec. III C, the construction of tight frames associated with the Weyl–Heisenberg group is essentially the same as that of tight frames associated with the $ax + b$ group. Tight frames associated with the $ax + b$ group were first introduced[13] in a different context closer to pure mathematics. In Ref. 13(b) one can find a definition of "quasiorthogonal families" very close to our tight frames, and a short discussion of the similarities between a "quasiorthogonal family" and an orthonormal basis. For the many miraculous properties of this orthonormal basis, see Ref. 14.

Finally, let us note that while we have restricted our discussion to $\mathscr{H} = L^2(\mathbf{R})$, it is possible to extend the discussion to $L^2(\mathbf{R}^n)$, as well for the Weyl–Heisenberg group as for the $ax + b$ group. In the latter case the unitary representation $U(\cdot)$ underlying the construction of frames, is no

longer irreducible. A more detailed analysis shows, however, that the essential feature is cyclicity of the representation rather than its irreducibility.[15]

II. THE WEYL–HEISENBERG CASE

A. Review of definitions and basic properties

The Weyl–Heisenberg group is the set $\mathbb{T}\times\mathbb{R}\times\mathbb{R}$ (where $\mathbb{T}$ is the set of complex numbers of modulus 1), with the group multiplication law

$$(z,q,p)(z',q',p') = (e^{i(pq'-p'q)/2}zz',q+q',p+p') .$$

We shall here be concerned with the irreducible unitary representation of this group acting in the Hilbert space $\mathscr{H} = L^2(\mathbb{R},dx)$, and given by

$$(W(z,q,p)f)(x) = ze^{-ipq/2}e^{ipx}f(x-q) .$$

The *Weyl operators* $W(q,p)$ are defined as

$$W(q,p) = W(1,q,p);$$

they satisfy the relations

$$W(q,p)W(q',p') = \exp[i(pq'-p'q)/2]W(q+q',p+p'),$$

an exponentiated form of the Heisenberg commutation relations. By a theorem of von Neumann, the above relations determine the irreducible family W up to unitary equivalence. A well-known property[16] of Weyl operators is the following; for all f_1,f_2,g_1,g_2, in $\mathscr{H}$ one has

$$\int\int dpdq\langle f_1,W(q,p)g_1\rangle\langle W(q,p)g_2,f_2\rangle = 2\pi\langle f_1|f_2\rangle\,\langle g_2|g_1\rangle. \tag{2.1}$$

Comparing this with (1.5) and (1.6), one sees that all elements of $\mathscr{H}$ are admissible and that $c_g = 2\pi\|g\|^2$. These two features are a consequence of the unimodularity of the Weyl–Heisenberg group.

The family of canonical coherent states is defined as a particular orbit under this set of unitary operators. The canonical coherent states can be defined as the family of vectors $W(q,p)\Omega$, where Ω is the ground state of the harmonic oscillator:

$$\Omega(x) = \pi^{-1/4}\exp(-x^2/2) .$$

One readily sees that this is equivalent to the customary definition of a canonical coherent state as the function

$$\pi^{-1/4}e^{-ipq/2}e^{ipx}\exp(-(x-q)^2/2) .$$

We shall often work with orbits of Weyl operators other than the canonical coherent states. We therefore introduce the notation

$$|q,p;g\rangle = W(q,p)g , \tag{2.2}$$

where g is any nonzero element of $\mathscr{H}$. The canonical coherent states are thus $|p,q;\Omega\rangle$. As a consequence of (2.1), one has

$$\int\int dqdp|q,p;g\rangle\,\langle q,p;g| = 2\pi\|g\|^2\mathbb{1} .$$

B. The von Neumann lattice and Zak transform

Take $a,b>0$. For any integer m,n, consider

$$|ma,nb;\Omega\rangle = W(ma,nb)\Omega . \tag{2.3}$$

It is known[8–10] that the linear span of the set $\{|ma,nb;\Omega\rangle;\ m,n \text{ in } \mathbb{Z}\}$ is dense in $\mathscr{H}$ if and only if $ab\leqslant 2\pi$. At the critical density $ab = 2\pi$, this set of points $\{(ma,nb)\}$ in phase space is called a von Neumann lattice.[7] In quantum mechanics, the associated set of canonical coherent states has a nice physical interpretation. It corresponds to choosing exactly one state per "semiclassical Gibbs cell," i.e., per cell of area h (Planck's constant).

Notice that the discrete set of Weyl operators $\{W(ma,n2\pi/a);\ m,n\in\mathbb{Z}\}$ is Abelian. This feature is exploited in the construction of the kq transform, or Zak transform,[17] which will turn out to be useful in what follows.

Denote by $\square$ the semiopen rectangle $\square = [-\pi/a,\pi/a)\times[-a/2,a/2)$.

The Zak transform is a unitary map from $L^2(\mathbb{R})$ onto $L^2([-\pi/a,\pi/a)\times[-a/2,a/2)) = L^2(\square)$ and is defined as follows. For a function f in $C_0^\infty(\mathbb{R})$ (infinitely differentiable functions with compact support), one defines its Zak transform Uf by

$$(Uf)(k,q) = \left(\frac{a}{2\pi}\right)^{1/2}\sum_l e^{ikal}f(q-la) , \tag{2.4}$$

where, for any q, only a finite number of terms in the sum contribute, due to the compactness of the support of f. The map U, defined by (2.4), is isometric from $C_0^\infty(\mathbb{R})\subset L^2(\mathbb{R})$ to $L^2(\square)$; there exists therefore an extension, which we shall also denote by U, to all of $L^2(\mathbb{R})$. It turns out that this extension maps $L^2(\mathbb{R})$ onto all of $L^2(\square)$; this is the Zak transform.

We ask now whether the family (2.3) constitutes a frame, i.e., whether the spectrum of the positive operator

$$P = \sum_m\sum_n|ma,n2\pi/a;\Omega\rangle\,\langle ma,n2\pi/a;\Omega|$$

is bounded away from zero. We shall see that the answer to this question is straightforward for the unitarily equivalent operator UPU^{-1}. The same technique was used in Ref. 10(a) to investigate the question whether the linear span of (2.3) and of similar families is dense.

An easy calculation leads to

$$[UW(ma,n2\pi/a)f](k,q) = e^{-ikma}e^{iqn2\pi/a}(Uf)(k,q) . \tag{2.5}$$

Hence, for $f\,L^2(\square)$, we have

$$\begin{aligned}\langle f,UPU^{-1}f\rangle &= \sum_m\sum_n\left|\int\int_\square dk\,dq\,e^{ikma}e^{iqn2\pi/a}(U\Omega)(k,q)f(k,q)\right|^2\\ &= \int dk\int dq|U\Omega(k,q)f(k,q)|^2 ,\end{aligned}$$

where we have used the basic unitary property of Fourier series expansions. This shows that the operator P is unitarily equivalent to multiplication by $|U\Omega(k,q)|^2$ in $L^2(\square)$. The spectrum of P is therefore exactly the numerical range of the function $|U\Omega(k,q)|^2$. The function $U\Omega$ is given by

$$U\Omega(k,q) = [a\pi^{-3/2}/2]^{1/2}\sum_l \exp[ikla - (q-la)^2/2]$$
$$= [a\pi^{-3/2}/2]^{1/2}\exp(-q^2/2)$$
$$\times\theta_3(a(k-iq)/2, \exp(-a^2/2)),$$

where θ_3 is one of Jacobi's theta functions[18]

$$\theta_3(z,u) = 1 + 2\sum_l u^{l^2}\cos(2lz).$$

The zeros of $U\Omega$ are therefore completely determined by the zeros of θ_3; one finds that the function $U\Omega$ has zeros at the corner of the semiopen rectangle $[-\pi/a,\pi/a) \times [-a/2,a/2)$, and nowhere else. (The fact that $U\Omega$ is zero at the corner can also be seen easily from its series expansion.) This is enough, however, to ensure that the spectrum of the multiplication operator by $|U\Omega(k,q)|^2$, and therefore also of P, contains zero. Therefore the family (2.3) associated to the von Neumann lattice is *not* a frame.

C. A frame of canonical coherent states

Since the family (2.3) with $ab = 2\pi$ is not a frame, it is clear that we have to look at lattices with higher density, i.e., with $ab < 2\pi$. [If $ab > 2\pi$, the linear span of the vectors (2.3) is not even dense.] The construction above, which uses the Zak transform, will not work for arbitrary a and b; if $b \neq 2\pi/a$, then Eq. (2.5) will no longer be true in general. This is due to the fact that in general the operators $W(ma,nb)$ do not mutually commute. It is, however, possible to use again the same construction in the case where the density is an integral multiple of the density for the von Neumann lattice. For the sake of definiteness, we shall consider the case where $ab = \pi$.

We now have to study the operator

$$P = \sum_m\sum_n \left|ma,\frac{n\pi}{a};\Omega\right\rangle\left\langle ma,\frac{n\pi}{a};\Omega\right|.$$

For $n = 2l$, it is clear from (2.3) that

$$|m,2l;\ a,\pi/a\rangle = W(ma,12\pi/a)\Omega = |ma,2\pi l/a;\Omega\rangle.$$

On the other hand,

$$|m,2l+1;\ a,\pi/a\rangle = e^{imab/4}W(ma,2\pi l/a)W(0,\pi/a)\Omega$$
$$= e^{imab/4}|ma,2\pi l/a;\ W(0,\pi/a)\Omega\rangle.$$

Hence

$$P = \sum_m\sum_n\left|ma,\frac{2\pi n}{a};\Omega\right\rangle\left\langle ma,\frac{2\pi n}{a};\Omega\right|$$
$$+\sum_m\sum_n\left|ma,\frac{2\pi n}{a};\ W\left(0,\frac{\pi}{a}\right)\Omega\right\rangle$$
$$\times\left\langle ma,\frac{2\pi n}{a};W\left(0,\frac{\pi}{a}\right)\Omega\right|.$$

Using (2.5) again, we then see that

$$\langle f,UPU^{-1}f\rangle = \int dk\int dq|f(k,q)|^2(|U\Omega(k,q)|^2$$
$$+|[UW(0,\pi/a)\Omega](k,q)|^2),$$

where U is again the Zak transform as defined above, in Sec. II B. A calculation of $UW(0,\pi/a)\Omega$ gives

$$[UW(0,\pi/a)\Omega](k,q)$$
$$= 2^{-1/2}\pi^{-3/4}a^{1/2}e^{i\pi q/a}\exp(-q^2/2)$$
$$\times\theta_3[(ak-aiq-\pi)/2, \exp(-a^2/2)],$$

hence

$$|U\Omega(k,q|^2 + |[UW(0,\pi/a)\Omega](k,q)|^2$$
$$= 2^{-1}\pi^{-3/2}\exp(-q^2)a$$
$$\times\{|\theta_3[a(k-iq)/2, \exp(-a^2/1)]|^2$$
$$+\theta_3[(ak-iqa-\pi)/2, \exp(-a^2/2)]|^2\}.$$

This function is continuous and has no zeros, since the zeros of $\theta_3[u,\exp(-a^2/2)]$ occur only at $u = \pi(m+\frac{1}{2}+ia^2(n+\frac{1}{2}))$. There exist therefore $A,B>0$ such that

$$A\leqslant|U\Omega(k,q)|^2 + |[UW(0,\pi/a)\Omega](k,q)|^2\leqslant B;$$

this implies that the set of canonical coherent states $\{|ma,n\pi/a;\ \Omega\rangle\}$ $(m,n\in\mathbf{Z})$ is a frame, with

$$A\leqslant\sum_m\sum_n\left|ma,\frac{n\pi}{a};\Omega\right\rangle\left\langle ma,\frac{n\pi}{a};\Omega\right|\leqslant B.$$

A numerical estimate of A and B gives, in the case $a = 2$,

$$A\geqslant 1.60,$$
$$B\leqslant 2.43.$$

Remark: The above analysis also works if the density of the chosen lattice is another, higher multiple of the critical von Neumann density, i.e., for $ab = 2\pi/n$, where $n = 3,4,\ldots$. The ratio B/A of the upper and lower bound of the frame is clearly a decreasing function of n.

D. Lattices with analyzing wavelets of compact support

We shall now consider families of the type $|na,mb;h\rangle$, where $h(x)$ is a function of compact support.

As an example, consider first the case where $h(x)$ is the characteristic function of an interval $[-L/2,L/2]$, i.e., $h(x) = 1$ if x belongs to this interval, and is zero otherwise. It is then easy to see that, with the choice $a = L$ and $b = 2\pi L$ (hence again $ab = 2\pi$), the family $\{|ma,nb;h\rangle\}$ $(m,n\in\mathbf{Z})$ consisting of the functions $\exp[2\pi inx/L]h(x)$ is an orthonormal basis of $L^2(\mathbf{R})$ and therefore certainly a frame.

For reasons explained in the Introduction, however, we prefer to work with smoother functions h. We shall see that under fairly general conditions, a lattice based on continuous functions of compact support also gives rise to a frame. The price to be paid is a higher density of the lattice; furthermore, the frame will not be tight in general.

Theorem 1: Let $h(x)$ be a continuous function on $\mathbf{R}$, with support in the interval $[-L/2,L/2]$. Assume that $h(x)$ is bounded away from zero in a subinterval $[-\mu L/2,\mu L/2]$ $(0<\mu<1)$:

$$|h(x)|\geqslant k,\quad \text{if}|x|\leqslant\mu L/2\quad(\mu<1).$$

Define now a lattice in phase space by taking $a = \mu L$ and $b = 2\pi/L$ (hence $ab = 2\pi\mu$, but the "oversampling parameter" μ^{-1} need not be an integer, contrary to Sec. II C). Consider the set of states

$$\{|ma,nb;h\rangle\}\quad(m,n\in\mathbf{Z});$$

then this set is a frame, with

$$A \geqslant L \inf_{|x|<\mu L/2} |h(x)|^2 \geqslant k$$

and

$$B \leqslant L \sup_{x\in\mathbf{R}} \left[\sum_m |h(x+m\mu)|^2\right]$$
$$\leqslant L(1+2[\mu^{-1}])(\|h\|_\infty)^2,$$

where $[\mu^{-1}]$ is the largest integer not exceeding μ^{-1}.

Proof: For typographical convenience, write $\Delta = [-L/2, L/2]$. Let f be any element of $L^2(\mathbf{R})$. Then

$$\langle ma,nb;h\,|\,f\rangle = e^{-imn\mu\pi} \int_\Delta dx\, h(x) \times e^{-2i\pi nx/L} f(x+m\mu L).$$

Hence, by considering the above integral as $L^{1/2}$ times the nth Fourier coefficient of the function $h(x)f(x+m\mu L)$ defined on the interval Δ,

$$\sum_n |\langle ma,nb;h\,|\,f\rangle|^2 = L\int_\Delta dx |h(x)|^2 |f(x+\mu mL)|^2$$
$$\geqslant k^2 L \int_{\mu\Delta} dx |f(x+\mu mL)|^2.$$

This implies

$$\sum_m \sum_n |\langle ma,nb;h\,|\,f\rangle|^2 \geqslant k\,|L \int dx |f(x)|^2.$$

On the other hand, we clearly have

$$\sum_m \sum_n |\langle ma,nb;h\,|\,f\rangle|^2$$
$$= L\int_\Delta dx \left(\sum_m |h(x+m\mu L)|^2\right) |f(x)|^2$$
$$\leqslant bL \int dx |f(x)|^2,$$

with

$$b = \sup_{x\in\mathbf{R}} \left[\sum_m |h(x+m\mu L)|^2\right]$$
$$\leqslant (2[\mu^{-1}]+1)(\|h\|_\infty)^2,$$

and so our assertions are proved.

E. Tight frames with analyzing wavelets of compact support

We keep the assumptions and notations of Sec. II D. The arguments of that subsection show that

$$\sum_n \left|\langle m\mu L, \frac{2\pi n}{L}; h\,|f\rangle\right|^2$$
$$= L\int_\Delta dx |h(x)|^2 |f(x+\mu mL)|^2$$
$$= L\int dx |h(x-\mu mL)|^2 |f(x)|^2.$$

Consequently we have

$$\sum_n \sum_m \left|\left\langle m\mu L, \frac{2\pi n}{L}; h\right\rangle f\right|^2$$
$$= L\int dx |f(x)|^2 \left[\sum_m |h(x+\mu mL)|^2\right],$$

and we obtain the following result.

Theorem 2: Let $h(x)$ be continuous on $\mathbf{R}$, with support in $[-L/2, L/2]$, and bounded away from zero on $[-\mu L/2, \mu L/2]$, where $0<\mu<1$. Assume furthermore that the function $\Sigma_m |h(x+\mu mL)|^2$ is a constant, i.e., independent of x. Then the family $\{|m\mu L, 2\pi n/L; h\rangle\}$ $(m,n\in\mathbf{Z})$ is a tight frame.

Remark: By the assumptions on h, the sum $\Sigma_m |h(x+\mu mL)|^2$ has only finitely many nonzero terms, and defines a continuous function of x.

We shall now give a procedure for constructing functions h that are k times continuously differentiable and satisfy the condition in Theorem 2:

$$\sum_m |h(x+m\mu L)|^2 = \text{const.} \tag{2.6}$$

Here k may be any positive integer or even ∞. We start by choosing a function g that is $2k$ times continuously differentiable and such that $g(x)=0$ for $x\leqslant 0$, and $g(x)=1$ for $x\geqslant 1$. Assume in addition that g is everywhere increasing.

For the sake of simplicity, we shall now assume that $\mu \geqslant \frac{1}{2}$. We then define h as follows:

$$h(x) = \begin{cases} 0, & \text{for } x \leqslant -L/2, \\ \{g[(x/L+1/2)/(1-\mu)]\}^{1/2}, & \\ \qquad \text{for } -L/2 \leqslant x \leqslant -L(2\mu-1)/2, & \\ 1, & \text{for } -L(2\mu-1)/2 \leqslant x \leqslant L(2\mu-1)/2, \\ \{1-g([x/L-(2\mu-1)/2]/(1-\mu))\}^{1/2}, & \\ \qquad \text{for } L(2\mu-1)/2 \leqslant x \leqslant L/2, & \\ 0, & \text{for } x \geqslant L/2. \end{cases}$$

The function $h(x)$ defined in this way is non-negative, with support $[-L/2, L/2]$, and equal to 1 on $[-(2\mu-1)L/2, (2\mu-1)L/2]$. Since g is a C^{2k} function, one sees that h is indeed a C^k function. The points $x = \pm(2\mu-1)L/2$, where h becomes constant, have been chosen so that their distance to the furthest edge of supp(f) is exactly μL. It is now easy to check that h fulfills the condition (2.6): for $|x| \leqslant (2\mu-1)L/2$, one has

$$\sum_m |h(x+m\mu L)|^2 = |h(x)|^2 = 1;$$

and for x in $[(2\mu-1)L/2, L/2]$, one has

$$\sum_m |h(x+m\mu L)|^2$$
$$= |h(x)|^2 + |h(x-\mu L)|^2$$
$$= 1 - g((x/L-(2\mu-1)/2)/(1-\mu))$$
$$+ g((x-\mu L)/L+1/2)/(1-\mu)) = 1.$$

For x outside $[-(2\mu-1)L/2, L/2]$ the result follows by simple translation. Hence

$$\sum_m |h(x+m\mu L)|^2 = 1,$$

which implies

$$\sum_m \sum_n \left| m\mu L, \frac{n2\pi}{L}; h \right\rangle \left\langle m\mu L, \frac{n2\pi}{L}; h \right| = L\mathbf{1},$$

and we have constructed a tight frame!

The above construction may be clarified by the following easy example.

Example: We define a function h satisfying the condition (2.6) as follows:

$$h(x) = \begin{cases} 0, & \text{if} |x| \geqslant \pi/2, \\ \cos x, & \text{if} |x| \leqslant \pi/2. \end{cases}$$

Hence $L = \pi$. We take $\mu = \frac{1}{2}$. Then (see also Fig. 1), with χ the characteristic function of the interval $[-\pi/2, \pi/2]$, one has

$$\sum_m |h(x + \mu m L)|^2 = \sum_m \cos^2\left(x + \frac{m\pi}{2}\right)\chi\left(x + \frac{m\pi}{2}\right)$$
$$= \cos^2 x + \sin^2 x = 1.$$

In this example, the corresponding function g is the function

$$g(x) = \begin{cases} 0, & \text{if } x \geqslant 0, \\ \sin^2(\pi x/2), & \text{if } 0 \leqslant x \leqslant 1, \\ 1, & \text{if } x \geqslant 1. \end{cases}$$

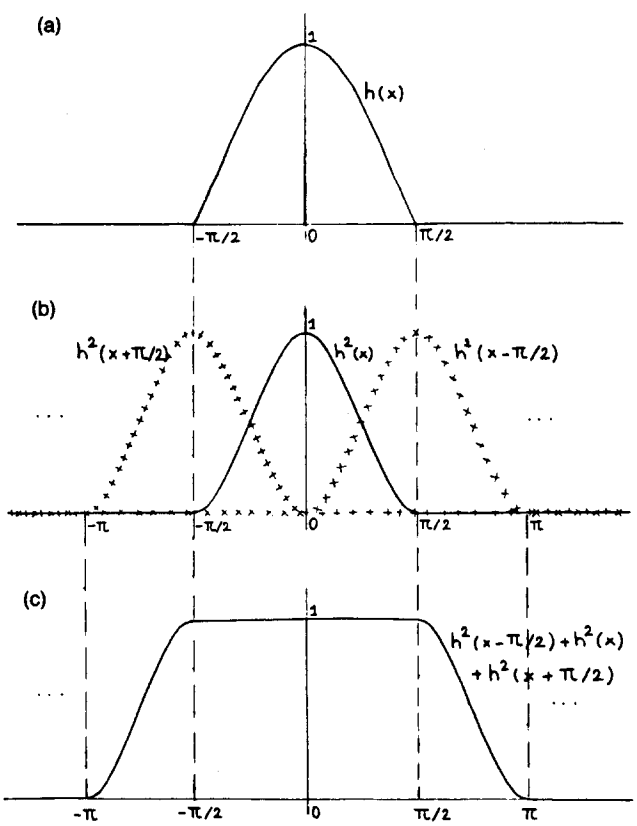

FIG. 1. (a) The function $h(x) = \cos x\, \chi_{[-\pi/2,\pi/2]}(x)$. (b) $h^2(x)$ and two translated copies, $h^2(x+\pi/2)$ and $h^2(x-\pi/2)$. (c) The sum $h^2(x-\pi/2) + h^2(x) + h^2(x+\pi/2)$ is equal to 1, for $-\pi/2 \leqslant x \leqslant \pi/2$. Analogously $\Sigma_{l=-N}^{N} h^2(x + l\pi/2) = 1$, for $-N\pi/2 \leqslant x \leqslant N\pi/2$, and $\Sigma_{l\in\mathbf{Z}} h^2(x + l\pi/2) = 1$, for all x.

Remark: In our construction, we have assumed that $\mu \geqslant \frac{1}{2}$. For smaller values of μ, a similar but more complicated construction can be made (see Appendix).

F. A closer look at condition (2.5)

Let h be a function continuous on $\mathbf{R}$, vanishing on the set $\mathbf{R} \setminus [-L/2, L/2]$ and nowhere else. The discussion of the preceding sections shows that the family of functions

$$h_{mn}(x) = e^{2i\pi nx/L} h(x + \mu m L) \quad (0 < \mu < 1; n, m \in \mathbf{Z})$$

is a tight frame if and only if (2.6) holds, i.e., one has

$$\sum_n f(x + na) = \text{const}, \tag{2.7}$$

with $a = \mu L$ and $f(x) = |h(x)|^2$.

In this subsection we shall study the class of functions that satisfy (2.7); at first, we shall not require f to be positive or to have compact support (as opposed to the assumptions on f in the preceding subsections). However, we need to impose some assumptions on f in order to ensure that the left-hand side of (2.7) is well defined. It will be convenient to work with the space $\mathscr{C}$ defined as follows.

Definition:

$$\mathscr{C} = \{f\colon \mathbf{R} \to \mathbf{C};\ f \text{ is measurable, and there exists a } C > 0 \text{ and a } K > 1, \text{ such that } |f(x)| \leqslant C(1 + |x|)^{-K}\}.$$

It is clear that, for f in $\mathscr{C}$, the series $\Sigma_n f(x + na)$ is absolutely convergent, uniformly on the interval $[-a/2, a/2]$. We shall now derive a necessary and sufficient condition for elements of $\mathscr{C}$ to satisfy (2.7). Take f in $\mathscr{C}$. Denote by Δ the interval $[-a/2, a/2]$. Define, for q in Δ,

$$F(q) = \sum_n f(q + na).$$

F is bounded, and hence belongs to $L^2(\Delta)$. We can therefore write its Fourier series as

$$F(q) = \sum_n c_n e^{2\pi i n q/a};$$

this series converges in the L^2 sense and also pointwise almost everywhere. The coefficients c_n are given by

$$c_n = \frac{1}{a} \int_\Delta dq\, e^{-2\pi i n q/a} \sum_n f(q + na)$$
$$= \frac{1}{a} \int dq\, e^{-2\pi i n q/a} f(q) = (2\pi)^{1/2} \frac{1}{a} \hat{f}\left(\frac{2\pi n}{a}\right),$$

where the interchange of integration and summation is justified, since the series converges absolutely. We thus have

$$\sum_n f(q + na) = (2\pi)^{1/2} \frac{1}{a} \sum_n \hat{f}\left(\frac{2\pi n}{a}\right) e^{2\pi i n q/a}. \tag{2.8}$$

This is Poisson's summation formula; see, e.g., Ref. 19, for a derivation of this formula for other classes of functions. It is now clear that the condition $\hat{f}(2\pi n/a) = 0$ for $n \neq 0$ is necessary and sufficient for $F = \text{const}$. Recapitulating, take f in $\mathscr{C}$. Then $\Sigma_n f(q + na)$ is independent of q if and only if, for every nonzero integer n, one has $\hat{f}(2\pi n/a) = 0$.

This motivates the following definition.

Definition:

$$\mathscr{I}_a = \{f\colon \mathbf{R}\to\mathbf{C};\ f\in\mathscr{C} \text{ and } \hat{f}(2\pi n/a) = 0,\ \text{for } n\in\mathbf{Z},\ n\neq 0\}.$$

The set $\mathscr{I}_a$ has many interesting properties. We enumerate a few of them.

(1) $\mathscr{I}_a$ is an ideal under convolution in $\mathbf{C}$, i.e., if $f\in\mathscr{I}_a$, $g\in\mathscr{C}$, then $f*g\in\mathscr{I}_a$.

(2) $\mathscr{I}_a$ is invariant under translations: if $f\in\mathscr{I}_a$ then, for every $y\in\mathbf{R}$, the function $x\to f(y-x)$ also belongs to $\mathscr{I}_a$.

(3) If $f\in\mathscr{I}_a$, then, for every $y>0$, the function $x\to f(yx)$ belongs to $\mathscr{I}_{ya}$.

(4) If $f\in\mathscr{I}_a$ then the integral of f can be replaced by a discrete sum:

$$\int dx\, f(x) = a\sum_n f(na) = a\sum_n f(q+na) \quad (\text{for all } q).$$

Proof:

(1) It is easy to check that for $f,g\in\mathscr{C}$, one has $f*g\in\mathscr{C}$. Since $\mathscr{C}\subset L^1(\mathbf{R})$, we have $(f*\hat{g})(k)\hat{g}(k)$ for all real k; hence $(f*\hat{g})(2\pi n/a) = 0$ if $\hat{f}(2n/a) = 0$.

The assertions (2) and (3) are trivial.

(4)
$$\int dx\, f(x) = (2\pi)^{1/2}\hat{f}(0) = a(2\pi)^{1/2}\frac{1}{a}\sum_n \hat{f}\left(\frac{2\pi n}{a}\right)e^{2\pi i n q/a},$$

since $\hat{f}(2\pi n/a) = 0$ for $n\neq 0$. Then (2.8) gives

$$\int dx\, f(x) = a\sum_n f(q+na).$$

Remark: Notice that (2.7) can be given an interpretation in terms of Zak transform, defined in Sec. II B. To impose condition (2.7) on a function f amounts to requiring that its Zak transform $(Uf)(k,q)$, defined on $[-\pi/a, \pi/a)\times[-a/2,a/2)$, should be constant along the line $k=0$.

III. THE AFFINE CASE

A. Review of definitions and basic properties

The group of shifts and dilations, or the "$ax+b$ group," is the set $\mathbf{R}^*\times\mathbf{R}$ (where $\mathbf{R}^*$ is the set of nonzero real numbers) with the group law

$$(a,b)(a',b') = (aa',ab'+b).$$

We shall here be concerned with the following representation of this group on $L^2(\mathbf{R})$:

$$[U(a,b)f](x) = |a|^{-1/2}f((x-b)/a). \tag{3.1}$$

This representation is irreducible and square integrable, so there exists a dense set of admissible vectors. The admissibility condition (1.5) can in this case be rewritten as

$$c_g = 2\pi\int dp|p|^{-1}|\hat{g}(p)|^2 < \infty, \tag{3.2}$$

where $\hat{g}$ is the Fourier transform of g:

$$\hat{g}(p) = (2\pi)^{-1/2}\int dx\, e^{-ipx}g(x)dx.$$

The fact that not every element of $L^2(\mathbf{R})$ is admissible with respect to the representation (3.1) stems from the nonunimodularity of the $ax+b$ group. The left-invariant measure on the $ax+b$ group is $a^{-2}\,da\,db$; the right-invariant measure is $|a|^{-1}\,da\,db$.

If g is an admissible vector, we define

$$|a,b;g\rangle = U(a,b)g;$$

such families of vectors can be called "affine coherent states."[11,12] The notation just used does not differ from the notation (2.2), used for the Weyl–Heisenberg group. However, it should be clear from the context which family is used at any one time. The general expression (1.4) in the Introduction can then be written for the $ax+b$ group in the following form:

$$\int a^{-2}\,da\,db\,|a,b;g\rangle\,\langle a,b;g| = c_g\mathbf{1},$$

where c_g is defined by (3.2).

B. Frames of affine coherent states, based on band-limited analyzing wavelets

The families that we shall consider are defined as

$$\{|a_n^+,b_{mn};g\rangle, |a_n^-,b_{mn};g\rangle\} \quad (m,n\in\mathbf{Z}),$$

where

$$a_n = \exp(\alpha n), \quad b_{mn} = \beta m a_n,$$

for some positive numbers α,β. We shall now derive restrictions on these numbers under which this discrete family is a frame.

The function g is supposed to be band limited, i.e., it is square integrable and its Fourier transform has compact support. We shall also assume that the support of $\hat{g}$ contains only strictly positive frequencies, i.e., is contained in an interval $[l,L]$, with $0<l<L<\infty$. This will enable us to decouple positive and negative frequencies in our calculations, which will turn out to be very convenient. Note that the requirement $l>0$ automatically guarantees that g is admissible [since the condition (3.2) is trivially satisfied].

Let f be any element of $L^2(\mathbf{R})$. We want to show that, under certain conditions on α,β,g to be derived here, we have

$$A\,\|f\|^2 \leq \sum_m\sum_n\{|\langle a_n^+,b_{nm};g|f\rangle|^2 + |\langle a_n^-,b_{mn};g|f\rangle|^2\} \leq B\,\|f\|^2,$$

with $A>0$, $B<\infty$.

An easy calculation leads to

$$\sum_m|\langle a_n^+,b_{mn};g|f\rangle|^2 = \frac{1}{a_n^+}\sum_m\left|\left(\int_l^L dw\, e^{-iw\beta m}\hat{g}(w)\hat{f}\left(\frac{w}{a_n^+}\right)\right)\right|^2.$$

If we impose on β the condition

$$\beta = 2\pi/(L-l),$$

this simplifies to

$$\sum_m |\langle a_n^+, b_{nm}; g|f\rangle|^2 = \frac{1}{a_n^+}\int dw \left|\hat{g}(w)|^2 \hat{f}\left(\frac{w}{a_n}\right)\right|^2$$
$$= \int dw |\hat{g}(a_n^+ w)|^2 . \qquad (3.3)$$

Define now

$$F_+(s) = f(e^s) ,$$
$$G(s) = \hat{g}(e^s) .$$

Since $a_n^+ = \exp(\alpha n) > 0$ for all $n \in \mathbf{Z}$, and since $\operatorname{supp} \hat{g} \subset \mathbf{R}_+$ we can make the substitution $t = e^s$ in the integral (3.3) and write

$$\sum_n \sum_m |\langle a_n^+, b_{mn}; g|f\rangle|^2$$
$$= \int ds \left[\sum_n |G(s+\alpha n)|^2\right] e^s |F+(s)|^2 . \qquad (3.4)$$

Since $\operatorname{supp} G = [\log l, \log L]$ is compact, only a finite number of terms contribute in the sum $\Sigma_n |G(s+\alpha n)|^2$ for any s. If we define now

$$A = \inf_{s\in\mathbf{R}} \left[\sum_n |G(s+\alpha n)|^2\right] ,$$
$$B = \sup_{s\in\mathbf{R}} \left[\sum_n |G(s+\alpha n)|^2\right] ,$$

then clearly

$$A \int_0^\infty dw |\hat{f}(w)|^2$$
$$< \sum_n \sum_m |\langle a_n^+, b_{mn}; g|f\rangle|^2 < B \int_0^\infty dw |\hat{f}(w)|^2 , \qquad (3.5)$$

where we have used

$$\int ds\, e^s |F_+(s)|^2 = \int_0^\infty dw |\hat{f}(w)|^2 .$$

A similar calculation can be made for vectors involving a_n^-. Introducing $F_-(s) = \hat{f}(-e^s)$, one finds

$$\sum_n \sum_m |\langle a_n^-, b_{mn}; g|f\rangle|^2$$
$$= \int ds \left[\sum_n |G(s+\alpha n)|^2\right] e^s |F_-(s)|^2 ,$$

hence

$$A \int_{-\infty}^0 dw |\hat{f}(w)|^2$$
$$< \sum_n \sum_m |\langle a_n^-, b_{mn}; g|f\rangle|^2 < B \int_{-\infty}^0 dw |\hat{f}(w)|^2 .$$

Combining this with (3.5) we find thus

$$A \|f\|^2 < \sum_n \sum_m \{|\langle a_n^+, b_{mn}; g|f\rangle|^2$$
$$+ |\langle a_n^-, b_{mn}; g|f\rangle|^2\} < B \|f\|^2 . \qquad (3.6)$$

If we can derive conditions on α, β, g, ensuring that $A > 0, B < \infty$, then (3.6) implies that under these conditions the set $\{|a^\pm, b_{mn}; g\rangle;\ m,n \in \mathbf{Z}\}$ is a frame. Since $\operatorname{supp} \hat{g} = [\log l, \log L]$, it is clear that A is zero unless $\alpha < \log(L/l)$. If we assume that $\hat{g}(w)$ is a continuous function without zeros in the interior of its support, then this condition is also sufficient to ensure that $A > 0$. Indeed, we then have

$$A \geq \inf\{|G(s)|^2;\ \log l + (\log(L/l) - \alpha)/2$$
$$\leq s \leq \log L (\log - (L/l) - \alpha)/2\} .$$

As for B, it is not hard to show that

$$B < \{2[\alpha^{-1}\log(L/l)] + 1\} \|\hat{g}\|_\infty^2 < \infty ,$$

where again we have used the notation $[\mu]$ for the largest integer not exceeding μ.

We have thus derived a set of sufficient conditions ensuring that our construction leads to a frame. The theorem below brings all these conditions together, rewritten in a slightly different form, and states our main conclusion.

Theorem: Let g: $\mathbf{R} \to \mathbf{C}$ satisfy the following conditions: (i) $\hat{g}$ has compact support $[l, kl]$, with $l > 0, k > 1$; and (ii) $|\hat{g}|$ is a continuous function, without zeros in the open interval (l, kl). Take $a \in (0, k)$. Define, for $m, n \in \mathbf{Z}$,

$$a_n^+ = \pm a^n ,$$
$$b_{mn} = 2\pi/[(k-1)l]\, m\, a^n .$$

Then the set $\{|a_n^\pm, b_{mn}; g\rangle;\ m,n \in \mathbf{Z}\}$ is a frame, i.e.,

$$A\mathbf{1} < \sum_n \sum_m \{|a_n^+, b_{mn}; g\rangle \langle a_n^+, b_{mn}; g|$$
$$+ |a_n^-, b_{mn}; g\rangle \langle a_n^-, b_{mn}; g|\} < B\mathbf{1} .$$

The lower and upper bounds A and B are given by

$$A = \inf_{w\in\mathbf{R}} \sum_n |\hat{g}(a^n w)|^2$$
$$\geq \inf\{|\hat{g}(w)|^2;\ w \in [l(k/a)^{1/2}, l(ka)^{1/2}]\} ,$$
$$B = \sup_{w\in R} \sum_n |\hat{g}(a^n w)|^2$$
$$< \{2[\log(k/a)] + 1\} \|\hat{g}\|_\infty^2 .$$

Remarks: (1) The same conclusions can be drawn under slightly less restrictive conditions on g. Strictly speaking we only need $\|\hat{g}\|_\infty < \infty$ and $\inf_{w\in\Delta} |\hat{g}(w)| > 0$ for any closed interval Δ contained in (l, kl); both these conditions are of course satisfied if $\hat{g}$ is continuous and has no zeros in (l, kl).

(2) As the calculations preceding the above theorem show, the positive and negative frequencies decouple neatly. It is therefore possible to choose a different function g_- (and accordingly, also a different lattice a_n, b_{mn}) for the negative frequency domain than for the positive frequency domain.

We have thus constructed a frame, based on a band-limited function g, under fairly general conditions on g. In general, the ratio B/A, comparing the upper with the lower bound, will be larger than 1. Again, however, as in the Weyl–Heisenberg case, it is possible to choose g in such a way that the frame becomes tight, i.e., $B/A = 1$; such tight frames have been used previously by one of us (Y. M.) in Ref. 13(a); they were also used in Ref. 13(b). The construction of such a frame follows more or less the same lines as in the Weyl–Heisenberg case (see Sec. II E); we shall show in the next subsection how the construction works in the present case.

C. Tight frames based on band-limited functions

We shall stick to the same construction as in the preceding subsection, and try to find a function g such that the frame based on g is quasiorthogonal.

Going back to (3.4), it is clear that the frame will be quasiorthogonal if and only if

$$\sum_n |G(s+\alpha n)|^2 = \text{const}, \tag{3.7}$$

where $G(s) = \hat{g}(e^s)$, and $\alpha = \log(a)$ with $a<k$, $\operatorname{supp}\hat{g} = [l,kl]$ $(l>0,\ k>1)$.

This condition (3.7) is exactly the same as the condition (2.6) in the Weyl–Heisenberg case; the analog of L is here $\log(kl) - \log(l) = \log(k)$, while the role of μ is played by $\alpha/\log k = \log(a)/\log(k) < 1$. The only difference is that the function G need not be centered around zero, as was supposed in Sec. II E.

We can therefore copy the construction made in Sec. II E to define a suitable G, hence a suitable g. Explicitly, and directly in terms of $\hat{g}$ rather than in terms of G, this gives

$$\hat{g}(w) = \begin{cases} 0, & \text{for } w<l, \\ [q(\log(w/l)/\log(k/a))]^{1/2}, & \\ \quad \text{for } l<w<lk/a, & \\ 1, & \text{for } lk/a<w<al \\ [1-q(\log(w/l)/\log(k/a))]^{1/2} & \\ \quad \text{for } al<w<kl, & \\ 0, & \text{for } w>kl, \end{cases} \tag{3.8}$$

where q is a C^{2k} function such that

$$q(x) = \begin{cases} 0, & \text{for } x<0, \\ 1, & \text{for } x>1, \end{cases} \tag{3.9}$$

a strictly increasing between 0 and 1.

Notice that we have assumed that $a^2>k$; this is equivalent to the assumption $\mu>\frac{1}{2}$ in Sec. II E. If $a^2<k$, a similar but more complicated construction can be made.

For $\hat{g}$ constructed as above, the condition (3.7) is satisfied;

$$\sum_n |G(s+\alpha n)|^2 = 1,$$

which implies that the corresponding frame is tight. We thus have proved the following theorem.

Theorem: Let the function $\hat{g}$, with compact support $[l,kl]$ $(l>0, k>1)$, be constructed according to (3.8), with $a>k^{1/2}$ where q is a function satisfying (3.9). Then the set of vectors

$$\{|a^n, 2\pi m a^n/(k-1)l; g\rangle,$$
$$|-a^n, 2\pi m a^n/(k-1)l; g\rangle;\ m,n\in\mathbf{Z}\}$$

(i.e., the set of functions

$$|a|^{-n/2} g[a^{-n}x + 2\pi m/(k-1)l],$$
$$|a|^{-n/2} g[-a^{-n}x + 2\pi m/(k-1)l])$$

is a quasiorthogonal frame in $L^2(\mathbf{R})$, with $A=B=1$. This means the following: If f is any function in $L^2(\mathbf{R})$ and if we define coefficients $f_{mn}^{(\pm)}$ $(m,n\in\mathbf{Z})$ by

$$f_{mn}^{(+)} = |a|^{-n/2} \int dx\, g\left[a^{-n}x + \frac{2\pi m}{(k-1)l}\right]^* f(x),$$

$$f_{mn}^{(-)} = |a|^{-n/2} \int dx\, g\left[-a^{-n}x + \frac{2\pi m}{(k-1)l}\right]^* f(x),$$

then

$$f(x) = \sum_m \sum_n f_{mm}^{(+)} |a|^{-n/2} g\left[a^{-n}x + \frac{2\pi m}{(k-1)l}\right] + \sum_m \sum_n f_{mm}^{(-)} |a|^{-n/2} g\left[-a^{-n}x + \frac{2\pi m}{(k-1)l}\right],$$

where the sum converges in $L^2(\mathbf{R})$.

Let us give some specific examples.

Example 1: We take $l=1$, $k=3$, $a=\sqrt{3}$. Define

$$\hat{g}(w) = \begin{cases} 0, & \text{for } w<1, \\ \sin[\pi \log w/\log 3], & \text{for } 1<w<3, \\ 0, & \text{for } w>3. \end{cases}$$

The corresponding g cannot be calculated in closed analytic form. A graph of $\operatorname{Re} g$, $\operatorname{Im} g$ is given in Fig. 2. The corresponding function q is the same as in the example in Sec. II E:

$$q(x) = \begin{cases} 0, & \text{for } x<0, \\ \sin^2(\pi x/2), & \text{for } 0<x<1, \\ 1, & \text{for } x>1. \end{cases}$$

Example 2: We take $l=1$, $k=4$, $a=2$. Define

$$\hat{g}(w) = \begin{cases} 0, & \text{for } w<1, \\ 2\sqrt{2}[\log w/\log 2]^2, & \text{for } 1<w<\sqrt{2}, \\ [1-8(1-\log w/\log 2)^4]^{1/2}, & \text{for } \sqrt{2}<w<2\sqrt{2}, \\ 2\sqrt{2}[2-(\log w/\log 2)]^2, & \text{for } 2\sqrt{2}<w<4, \\ 0, & \text{for } w>4. \end{cases}$$

The corresponding function q is

$$q(x) = \begin{cases} 0, & \text{for } x<0, \\ 8x^4, & \text{for } 0<x<\frac{1}{2}, \\ 1-8(1-x)^4, & \text{for } \frac{1}{2}<x<1, \\ 1, & \text{for } x>1. \end{cases}$$

Graphs of $\operatorname{Re} g$, $\operatorname{Im} g$ are given in Fig. 3.

Because of the correspondence, noted above, between tight frames for the $ax+b$ group and the Weyl–Heisenberg group, all the examples given in the Appendix for the Weyl–Heisenberg group can easily be transposed to the present case.

D. A closer look at the necessary and sufficient condition

The necessary and sufficient condition that a band-limited function g, concentrated on positive frequencies, has to satisfy in order to generate a tight frame, is given by (3.7). This can be rewritten as

$$\sum_n f(a^n w) = \text{const}, \tag{3.10}$$

where $f(w) = |\hat{g}(w)|^2$.

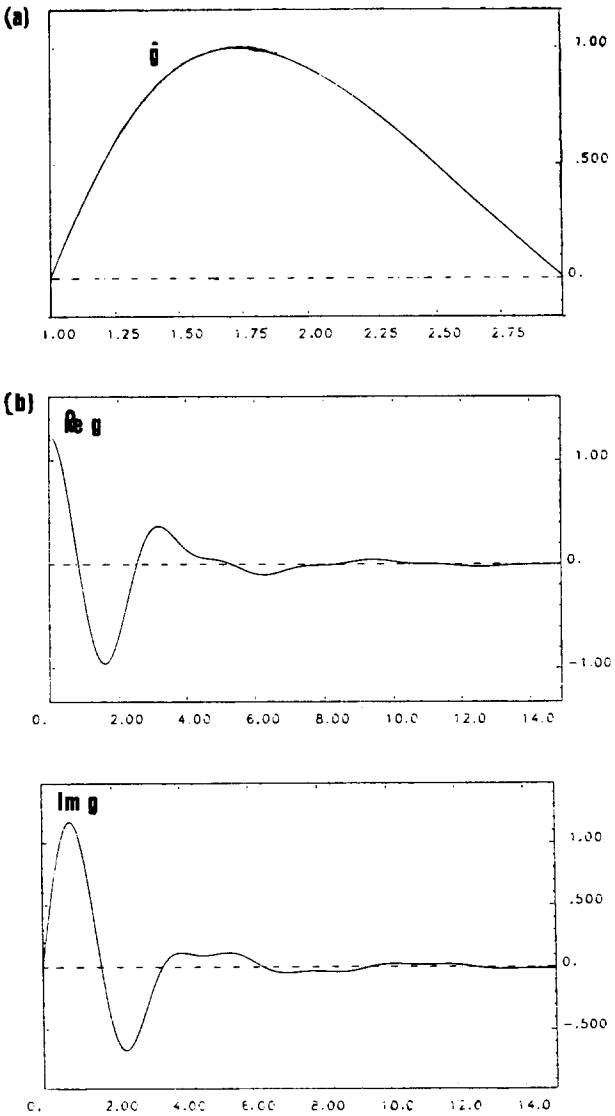

FIG. 2. (a) The function $\hat{g}(w) = \sin(\pi \log w/\log 3)\chi_{[1,3]}(w)$. (b) Real and imaginary parts of the inverse Fourier transform g of $\hat{g}$.

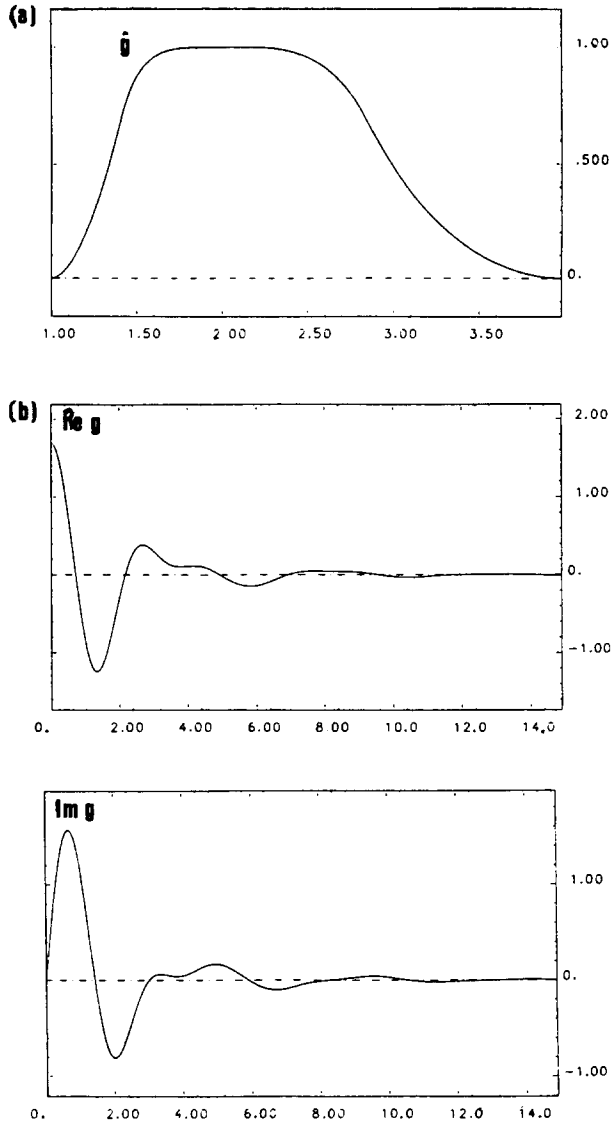

FIG. 3. (a) The function $\hat{g}(w)$ defined in example 2 of Sec. III D. (b) Real and imaginary parts of the inverse Fourier transform g or $\hat{g}$.

In this subsection we shall study the class of functions f: $\mathbf{R}_+ \to \mathbf{C}$, satisfying (3.10); for the purpose of this subsection only, we shall not require f to be positive or to have compact support. This study will be completely analogous to our study in Sec. II F of the functions satisfying condition (2.7); since the arguments run along exactly the same lines, we shall not go into as much detail here. The main difference is that we shall work with the Mellin transform of f rather than with its Fourier transform; this, of course, is due to the difference between (3.10), where the constant enters multiplicatively, and (2.7) where it enters additively.

The Mellin transform F of f: $\mathbf{R}_+ \to \mathbf{C}$ is defined by

$$F(s) = \int_0^\infty w^{s-1} f(w)\,dw\,;$$

the inversion formula is

$$f(w) = \frac{1}{2\pi i}\int ds\, t^{-s} F(s)\,,$$

where the integral is taken from $c - i\infty$ to $c + i\infty$ and where $c > 0$ has to be chosen so that the integral converges.

For the sake of convenience we shall restrict ourselves to functions $f \in \mathscr{C}^{\sim}$, where $\mathscr{C}^{\sim}$ is defined below.

Definition:

$$\mathscr{C}^{\sim} = \{f\colon \mathbf{R}_+ \to \mathbf{C};\ f \text{ is measurable and there exists } C > 0 \text{ and } k > 1 \text{ such that } |f(w)| < C(1 + |\log w|)^{-k}\}\,.$$

For functions $f \in \mathscr{C}^{\sim}$ the series $\Sigma_n f(a^n t)$ is absolutely convergent, uniformly on $(1,a]$; the Mellin transform F of f is well defined for purely imaginary arguments, and the inversion formula applies, with $c = 0$.

Notice that if we define g: $\mathbf{R} \to \mathbf{C}$ by $g(x) = f(e^x)$, we immediately find

$$f \in \mathscr{C}^{\sim} \Leftrightarrow g \in \mathscr{C}$$

(where $\mathscr{C}$ is defined in Sec. II F) and

$$F(ik) = (2\pi)^{1/2}\hat{g}(k) \quad (k \in \mathbf{R})\,.$$

This enables us to translate the results of Sec. II F to the present situation.

We have, for $f \in \mathscr{C}^{\sim}$,

$$\sum_n f(a^n w) = \frac{1}{\log a} \sum_n F\left(\frac{2\pi i n}{\log a}\right) w^{2\pi i n/\log a},$$

which leads to the following theorem.

Theorem: Take $f \in \mathscr{C}^{\sim}$. Then $\Sigma_n f(a^n w)$ is independent of w if and only if $F(2\pi i n/\log a) = 0$ for all nonzero integers n.

This then motivates the following definition.

Definition:

$$\mathscr{I}_a^{\sim} = \{f\colon \mathbf{R}_+ \to \mathbf{C}, \text{ and } F(2\pi i n/\log a) = 0, \quad \text{for } n \in \mathbf{Z},\ n \neq 0\}.$$

The following theorem lists a few properties of $\mathscr{I}_a^{\sim}$.

Theorem: (1) $\mathscr{I}_a^{\sim}$ is an ideal in $\mathscr{C}^{\sim}$ under "Mellin convolution," i.e., if $f \in \mathscr{I}_a^{\sim}$, $g \in \mathscr{C}^{\sim}$, then the function $f \underset{M}{*} g$ defined by

$$(f \underset{M}{*} g)(w) = \int_0^\infty \frac{du}{u} f(u) g\left(\frac{t}{u}\right),$$

belongs to $\mathscr{I}_a^{\sim}$.

(2) $\mathscr{I}_a^{\sim}$ is invariant under dilations: if $f \in \mathscr{I}_a^{\sim}$, then, for every $u \neq 0$, the function $w \to f(uw)$ also belongs to $\mathscr{I}_a^{\sim}$.

(3) If $f \in \mathscr{I}_a$, then the integral of $w^{-1} f(w)$ can be replaced by a discrete sum:

$$\int_0^\infty dw\, w^{-1} f(w) = \log a \sum_n f(a^n)$$

$$= \log a \sum_n f(a^n t), \quad \text{for all } t.$$

Notice that the "Mellin convolution" defined above is exactly the convolution in the sense of Ref. 19, with respect to the multiplicative group of nonzero real numbers. It is obvious that the Mellin transform of a "Mellin convolution" of two functions is given, up to a constant factor, by the product of the two Mellin transforms.

ACKNOWLEDGMENT

One of us (I. D.) would like to thank the Centre de Physique Theorique 2, CNRS-Marseille, where part of this work was done, for its hospitality and support.

APPENDIX: MORE EXAMPLES

We give a few more examples of functions h supported in $[-L/2, L/2]$ and satisfying the condition

$$\sum_{n \in \mathbf{Z}} |h(x + n\mu L)|^2 = \text{const}. \tag{A1}$$

We start by showing how one can extend the construction given in Sec. II E, for the case $\mu \geqslant \frac{1}{2}$, to more general μ.

For $2^{-(k+1)} \leqslant \mu \leqslant 2^{-k}$, with $k \geqslant 0$, one can always define $\mu^{\sim} = 2^k \mu$. Obviously $\frac{1}{2} \leqslant \mu^{\sim} \leqslant 1$. If we replace μ by $\mu^{\sim}$ in the construction of Sec. II E, we obtain a function h satisfying

$$\sum_{n \in \mathbf{Z}} |h(x + n 2^k \mu L)|^2 = \text{const} = C.$$

Hence

$$\sum_{n \in \mathbf{Z}} |h(x + n\mu L)|^2 = 2^k C.$$

Functions h constructed in this way thus obviously satisfy condition (A1). It is clear from the construction that the tight frame $\{|n\mu L, m 2\pi/L; h\rangle\}$ generated by h is in this case a superposition of the tight frame $\{|n\mu^{\sim} L, m 2\pi/L; h\rangle\}$ and translated copies of this frame.

There also exist, of course, functions h satisfying (A1), for $\mu < \frac{1}{2}$, which cannot be reduced to the case $\frac{1}{2} \leqslant \mu \leqslant 1$. We give here an example of such a function, for the case $\mu = \frac{1}{3}$.

Let g be again a C^{2k} function, strictly increasing, such that

$$g(x) = \begin{cases} 0, & \text{for } x \leqslant 0, \\ 1, & \text{for } x \geqslant 1. \end{cases}$$

Define then

$$h(x) = \begin{cases} 0, & \text{for } x \leqslant -L/2, \\ \{g[(6x+3L)/4L]\}^{1/2}, & \text{for } -L/2 \leqslant x \leqslant L/6, \\ \{1 + g(1/2) - g[(6x+L)/4L] \\ \qquad - g[(6x-L)/4L]\}^{1/2}, \\ \qquad \text{for } L/6 \leqslant x \leqslant L/2, \\ 0, & \text{for } x \geqslant L/2. \end{cases}$$

Obviously h has support $[-L/2, L/2]$. One can check that h is a C^k function satisfying

$$\sum_{n \in \mathbf{Z}} \left| h\left(x + \frac{nL}{3}\right)\right|^2 = 1 + g\left(\frac{1}{2}\right).$$

This example can also be adapted to cover the case $\frac{1}{2} \geqslant \mu \geqslant \frac{1}{3}$ (instead of only $\mu = \frac{1}{3}$).

Finally, note that another class of examples, for the special cases $\mu = 1/2(k+1)$, with k a positive integer, can be constructed with the help of spline functions. Choose a knot sequence $t = (t_j)_{j \in \mathbf{Z}}$ with equidistant knots, $t_{j+1} - t_j = d > 0$ for all j. Let $B_{j,2k+2,t}$ be the jth B-spline of order $2k+2$ for the knot sequence t. (For the definition of B-splines, see, e.g., Ref. 20.) Then the $B_{j,2k+2,t}$ are all translated copies of $B_{0,2k+2,t}$:

$$B_j(x) = B_0(x - jd).$$

The B_j are (positive) C^{2k} functions, with support $[t_j, t_{j+2(k+1)}] = [t_j, t_j + 2(k+1)d]$. They have, moreover, the property that

$$\sum_{j \in \mathbf{Z}} B_j(x) = 1, \quad \text{for all } x$$

(only a finite number of terms contribute for any x). It is now easy to check that the function h, defined by

$$h(x) = \{B_{0,k,t}[t_0 + (k+1)d(2x+L)/L]\}^{1/2},$$

is a C^k function with support $[-L/2, L/2]$, satisfying condition (A1) with $\mu = 1/2(k+1)$.

[1]R. M. Young, *An Introduction to Non-harmonic Fourier Series* (Academic, New York, 1980).

[2]R. J. Duffin and A. C. Schaeffer, Trans. Am. Math. Soc. 72, 341 (1952).

[3]I. Daubechies and A. Grossmann "Frames in the Bargmann space of entire functions" (to be published).
[4]A. Grossmann and J. Morlet, SIAM J. Math. Ann. **15**, 723 (1984); A. Grossmann and J. Morlet, "Decomposition of functions into wavelets of constant shape, and related transforms," to appear in *Mathematics and Physics, Lectures on Recent Results* (World Scientific, Singapore, 1985); P. Goupillaud, A. Grossmann, and J. Morlet, Geoexploration **23**, 85 (1984).
[5]M. Bertero, C. De Mol, and G. Viano, "The stability of inverse problems," in *Inverse Scattering Problems in Optics*, edited by H. P. Baltes (Springer, Berlin, 1980).
[6](a) M. Duflo and C. C. Moore, J. Funct. Anal. **21**, 208 (1976); A. L. Carey, Bull. Austr. Math. Soc. **15**, 1 (1976); (b) S. A. Gaal, *Linear Analysis and Representation Theory* (Springer, Berlin, 1973); (c) A. Grossmann, J. Morlet, and T. Paul, J. Math. Phys. **26**, 2473 (1985).
[7]J. von Neumann, *Mathematical Foundation of Quantum Mechanics* (Princeton U. P., Princeton, NJ, 1985) (English translation).
[8]V. Bargmann, P. Butera, L. Girardello, and J. R. Klauder, Rep. Math. Phys. **2**, 221 (1971).
[9]A. M. Perelomov, Teor. Mat. Fiz. **6**, 213 (1971).
[10](a) H. Bacry, A. Grossmann, and J. Zak, Phys. Rev. B **12**, 1118 (1975); (b) R. Balian, C. R. Acad. Sci. Paris, Ser. 2 **292**, 1357 (1981).
[11]E. W. Aslaksen and J. R. Klauder, J. Math. Phys. **10**, 2267 (1969).
[12]T. Paul, J. Math. Phys. **25**, 3252 (1984); A. Grossmann and T. Paul, "Wave functions on subgroups of the group of affine canonical transformations," in *Lecture Notes in Physics*, Vol. 211 (Springer, Berlin, 1984).
[13](a) Y. Meyer, "La transformation en ondellettes et les nouveaux paraproduits," to be published; (b) M. Frasier and B. Jawerth, "Decomposition of Besov spaces," to be published.
[14]Y. Meyer, "La transformation en ondelettes: de la recherche pétrolière à la géométrie des espaces de Banach," to be published; P. G. Lemarié and Y. Meyer, "Ondelettes et bases hilbertiennes," to be published.
[15]A. Grossmann and T. Paul (in preparation).
[16]J. Klauder and B. S. Skagerstam, *Coherent States. Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
[17]J. Zak, in *Solid State Physics*, edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1972), Vol. 27.
[18]M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).
[19]E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, II* (Springer, Berlin, 1970).
[20]C. de Boor, *A Practical Guide to Splines* (Springer, Berlin, 1978).