The SVD of a circulant matrix

Let H be an $n \times n$ circular convolution operator (assume n is even). That is, the action Hf of H on a signal $f \in \mathbb{R}^n$ is

$$Hf = f \circledast h,$$

where $\circledast$ indicates circular convolution, and $h \in \mathbb{R}^n$ is the first row of H. We know that the DFT diagonalizes H; that is, we can write

$$H = F^{-1}\hat{H}F,$$

where F is the standard DFT matrix,

$$F_{k,t} = e^{-j2\pi(k-1)(t-1)/n} \quad \text{for} \quad t,k = 1,\ldots,n,$$

and $\hat{H}$ is a diagonal matrix with $\hat{h} = Fh$ along the diagonal:

$$\hat{H} = \mathrm{diag}(\hat{h}).$$

We may also write this as

$$H = \tilde{F}^*\hat{H}\tilde{F}, \quad \text{where} \quad \tilde{F} = \frac{1}{\sqrt{n}}F.$$

This is not quite an SVD, since the entries in $\tilde{F}$ and $\hat{H}$ are complex.

We can, however, quickly calculate the SVD from the DFT of h. By exploiting the symmetries in the Fourier transform and in $\hat{h}$ (i.e. the fact that is conjugate symmetric and so $\hat{h}_{n-k+2} = \hat{h}_k^*$ for $k = 2,\ldots,n/2$ and $\hat{h}_1$ and $\hat{h}_{n/2+1}$ are real), we can write

$$H = U\Lambda V^T$$

where the entires of V and U are

$$V_{m,k} = \begin{cases} \frac{1}{\sqrt{n}} & k = 1 \\ \sqrt{\frac{2}{n}}\cos\left(\frac{2\pi(k-1)(m-1)}{n}\right) & k = 2,\ldots,n/2 \\ \sqrt{\frac{1}{n}}(-1)^{m-1} & k = n/2+1 \\ \sqrt{\frac{2}{n}}\sin\left(\frac{2\pi(k-1)(m-1)}{n}\right) & k = n/2+2,\ldots,n \end{cases}$$

$$U_{m,k} = \begin{cases} \frac{z_1}{\sqrt{n}} & k = 1 \\ \sqrt{\frac{2}{n}}\cos\left(\frac{2\pi(k-1)(m-1)}{n} + \theta_k\right) & k = 2,\ldots,n/2 \\ \frac{z_{n/2+1}}{\sqrt{n}}(-1)^{m-1} & k = n/2+1 \\ \sqrt{\frac{2}{n}}\sin\left(\frac{2\pi(k-1)(m-1)}{n} + \theta_k\right) & k = n/2+2,\ldots,n \end{cases}$$

with

$$z_1 = \text{sgn}(\hat{h}_1)$$
$$\theta_k = \angle \hat{h}_k$$
$$z_{n/2+1} = \text{sgn}(\hat{h}_{n/2+1}),$$

and the singular values are the magnitudes of the DFT of h:

$$\Lambda_{k,k} = |\hat{h}_k|.$$

Notice that the phases/signs of $\hat{h}$ are being absorbed into the left singular vectors U. Also, the singular values will not be sorted in decreasing order with these definitions.

The following MATLAB code implements this SVD.

```
function [U, Lambda, V] = conv_svd(h)

n = length(h);
hhat = fft(h(:));

V = zeros(n);
V(:,1) = sqrt(1/n)*ones(n,1);
V(:,2:n/2) = sqrt(2/n)*cos(2*pi/n*(0:n-1)'*(1:n/2-1));
V(:,n/2+1) = sqrt(1/n)*(-1).^((0:n-1)');
V(:,n/2+2:n) = sqrt(2/n)*sin(2*pi/n*(0:n-1)'*(n/2+1:n-1));

U = zeros(n);
U(:,1) = sqrt(1/n)*sign(hhat(1))*ones(n,1);
U(:,2:n/2) = sqrt(2/n)*cos(2*pi/n*(0:n-1)'*(1:n/2-1) + ones(n,1)*angle(hhat(2:n/2))');
U(:,n/2+1) = sqrt(1/n)*sign(hhat(n/2+1))*(-1).^((0:n-1)');
U(:,n/2+2:n) = sqrt(2/n)*sin(2*pi/n*(0:n-1)'*(n/2+1:n-1) + ones(n,1)*angle(hhat(n/2+2:n))');

Lambda = diag(abs(hhat));
```