II Time-Frequency Frames & Wavelets

We are now very familiar with the idea of decomposing a signal using an <u>orthobasis</u> or a <u>frame</u>.

Now we will take a quick look at <u>constructing</u> bases & frames that are <u>local</u> in both <u>time</u> & <u>frequency</u>.

Why? Interesting signals (pretty much anything having to do with waves propagating) often have <u>simple characterizations</u> in terms of their time/frequency content.
→ this comes in very handy when it comes time to process these signals (much, much more on this later)

This is essentially the same overview as in Mallat 5.4 & 8.3-8.5
For depth, read the classic papers
- Daubechies, Grossman, Meyer: "Painless nonorthogonal expansions" (1986)
- Daubechies: "The wavelet transform, time-frequency localization, and signal analysis" (1990)

Also, Daubechies' book: <u>Ten Lectures on Wavelets</u>.

First shot:

Block Bases

Construct an $\perp$-basis for $L_2(\mathbb{R})$ in the following manner:

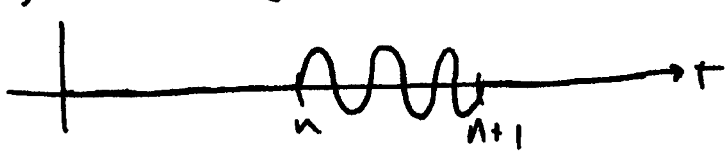
① Divide up the real line into intervals

$$[n, n+1] \qquad n \in \mathbb{Z}$$

② Calculate the Fourier Series on each interval

Let $$g_n(t) = \begin{cases} 1 & n \le t \le n+1 \\ 0 & \text{otherwise} \end{cases}$$

Set $$\phi_{n,k}(t) = g_n(t) \cdot e^{j2\pi k t}$$



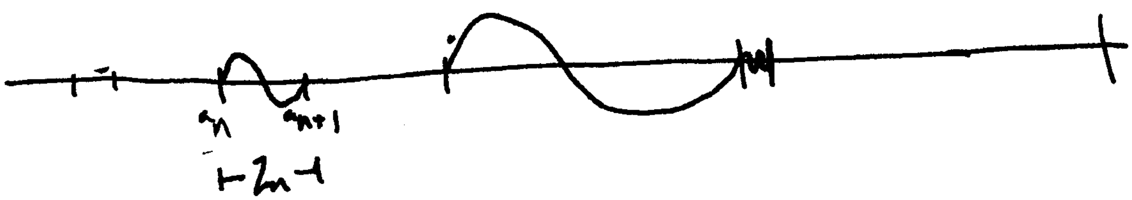
Then $\{\phi_{n,k}\}_{n,k \in \mathbb{Z}}$ is an orthobasis for $L_2(\mathbb{R})$

proof:

In fact, we can partition the real line arbitrarily with a sequence $\{a_n\}_{n\in\mathbb{Z}}$

$$\lim_{n\to-\infty} a_n = -\infty \qquad \lim_{n\to+\infty} a_n = +\infty$$

and use F.S. harmonics proportional to the length

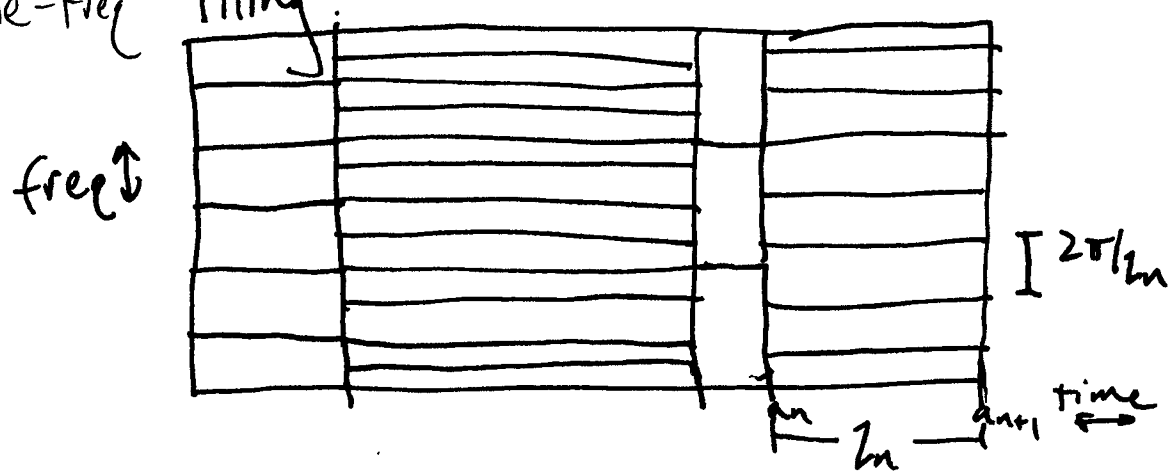


set $Z_n = a_{n+1} - a_n$

$$g_n(t) = \begin{cases} 1 & a_n \le t \le a_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_{n,k}(t) = \frac{1}{\sqrt{Z_n}}\, g_n(t) \cdot e^{j\frac{2\pi k t}{Z_n}}$$

Time-Freq tiling:



Note: width × height = 2π

Is this a good orthobasis for $L_2(\mathbb{R})$?

In practice, <u>no</u> (unfortunate).

Why not?

<u>Interlude: Fourier Transforms & smoothness</u>

$f(t) \in L_2(\mathbb{R})$ Fourier transform $\hat{f}(\omega) = \int f(t) e^{-j\omega t} dt$

The <u>asymptotic decay</u> of $\hat{f}(\omega)$ tells us about the <u>smoothness</u> of $f(t)$.

First, if $\hat{f}(\omega) \in L_1(\mathbb{R})$, then

$$|f(t)| = \frac{1}{2\pi}\left|\int \hat{f}(\omega) e^{j\omega t} d\omega\right| \leq \frac{1}{2\pi}\int |\hat{f}(\omega) e^{j\omega t}| d\omega$$

$$\leq \frac{1}{2\pi}\int |\hat{f}(\omega)| d\omega < +\infty$$

$\Rightarrow$ $f(t)$ is <u>bounded</u> (easy to see) and <u>continuous</u> (requires a short argument)

Now, suppose

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)| \left(1 + |\omega|^p\right) d\omega < +\infty$$

which is true if

$$|\hat{f}(\omega)| \leq \frac{C}{1 + |\omega|^{p+1+\varepsilon}} \quad \text{for some } C \text{ and } \varepsilon > 0$$

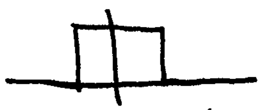
Then

$$|f^{(k)}(t)| \le \int_{-\infty}^{\infty} |\hat{f}(\omega)| \cdot |\omega|^k \, d\omega$$

$$< \infty$$

$\Rightarrow$ all derivatives up to order p are bounded and continuous.

<u>But</u> it only takes a <u>single</u> discontinuity to mess things up.

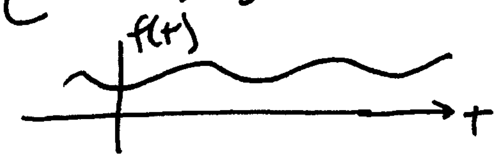
$f(t) =$  (piecewise constant)

how does $\hat{f}(\omega)$ decay ?

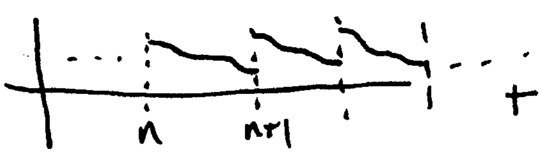
Returning to our example

$$g_n(t) = \begin{cases} 1 & n \le t \le n+1 \\ 0 & \text{otherwise} \end{cases} \qquad \theta_{n,k}(t) = g_n(t)\, e^{j 2\pi k t}$$

Say $f \in C^\infty \longrightarrow$ smooth as can be



How do the $\langle f, \theta_{n,k} \rangle$ behave as k gets large?



The problem is that the windows $g_n(t)$ are not really local in frequency (slow decay).

Time-Frequency frames

Let's generalize the process above, so we can consider smooth windows.

Given a window function $g(t)$, let

$$g_{u,\gamma}(t) = g(\underbrace{t-u}_{\text{shift in time}})\, e^{j\overbrace{\gamma t}^{\text{modulation in frequency}}}$$

We will assume throughout that

$$\|g\|_{L_2} = \|g_{u,\gamma}\|_{L_2} = 1$$

The <u>windowed Fourier transform</u> is a function on $\mathbb{R}\times\mathbb{R}$:

$$[Sf](u,\gamma) = \{\langle f, g_{u,\gamma}\rangle : u,\gamma \in \mathbb{R}\}$$

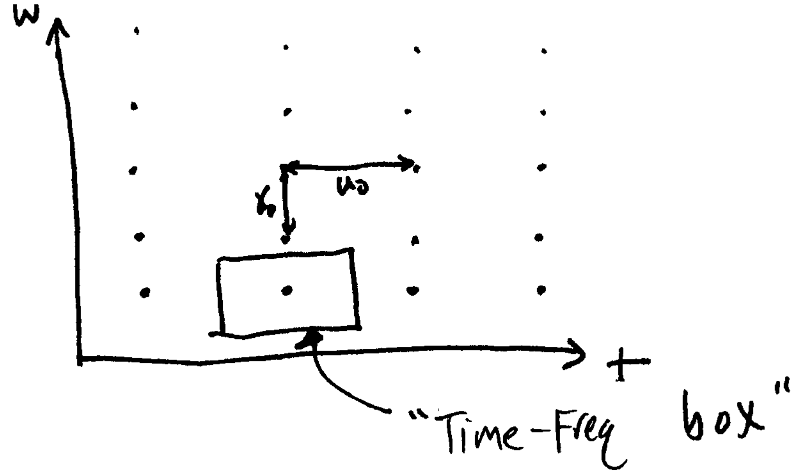
We will look at <u>samplings</u> of $Sf(u,\gamma)$

$$\{\langle f, g_{u_n,\gamma_k}\rangle : n,k \in \mathbb{Z}\}$$

and in fact will consider only <u>uniform samplings</u>

$$g_{n,k}(t) = g(t - n\cdot u_0)\, e^{jk\gamma_0 t}$$

u_0 = spacing in time
γ_0 = spacing in frequency



We ask:

① Under what conditions is $\{g_{n,k}(t)\}_{n,k\in\mathbb{Z}}$ a frame?

② Can we find a better T-F $\perp$-basis than when $g(t)=$?

Necessary conditions for a frame (Daubechies)

For $\{g_{n,k}(t)\}_{n,k\in\mathbb{Z}}$ to be a frame, we need

$$\frac{2\pi}{u_0\gamma_0} \geq 1 \qquad (u_0\gamma_0 \leq 2\pi),$$

and the frame bounds A, B will satisfy

ⓐ $$A \leq \frac{2\pi}{u_0\gamma_0} \leq B$$

ⓑ $$A \leq \frac{2\pi}{\gamma_0}\cdot\sum_{n=-\infty}^{\infty} |g(t-nu_0)|^2 \leq B$$

ⓒ $$A \leq \frac{1}{u_0}\cdot\sum_{k=-\infty}^{\infty} |\hat{g}(w-k\gamma_0)|^2 \leq B$$

At the very least, the windows need to cover both the time & frequency axis

What is A when $\{g(t-nu_0)\}$ looks like

can put stuff here that won't be "seen"

The sufficient conditions are more involved, but have much the same gist.

Moral: We have a frame if the sampling is dense enough, but not so dense that it clusters around isolated points.

Dual Frame: Interestingly, the dual frame

$$\tilde{g}_{n,k} = (\Phi^*\Phi)^{-1} g_{n,k}$$

are also windowed sinusoids:

$$\tilde{g}_{n,k}(t) = \tilde{g}(t-nu_0)\cdot e^{jk\gamma_0 t}$$

same spacing

with

$$\tilde{g} = (\Phi^*\Phi)^{-1} g$$

See Mallat for proof (not that hard).

The bad news about orthobases

Since all the $\|g_{n,k}\|_{L_2} = 1$, we will need

$$A = B = 1 \quad \Rightarrow \quad u_0 \gamma_0 = 2\pi$$

unfortunately....

Balian-Low Theorem

If $\{g_{n,k}\}_{n,k \in \mathbb{Z}}$ is a windowed Fourier frame with $u_0 \gamma_0 = 2\pi$, then either

$$\int_{-\infty}^{\infty} t^2 |g(t)|^2 \, dt = +\infty \quad \text{-or-} \quad \int_{-\infty}^{\infty} \omega^2 |\hat{g}(\omega)|^2 \, d\omega = +\infty$$

g is either non-local in time -or- non-local in frequency (smooth)

$\rightarrow$ $g(t)$ cannot be compact in time <u>and</u> differentiable.

So, T-F orthobases are not that great.

But we can do wonderful things w/ tight and almost-tight <u>frames</u>.

Gaussian window (Gabor frames)

$$g(t) = \pi^{-1/4} e^{-t^2/2} \longleftrightarrow \hat{g}(\omega) = C \cdot e^{-\omega^2/2}$$

$$(C = \sqrt{2}\,\pi^{1/4})$$

very smooth, very concentrated in both time and frequency

Of course, we will need $\underline{u_0 \gamma_0 < 2\pi}$ and all $g_{n,k}$ w/ this condition are frames

but the frame bound are very nice for modest oversampling

In this table, $A_0 \le A < B \le B_0$

we can compute bounds on the frame bounds

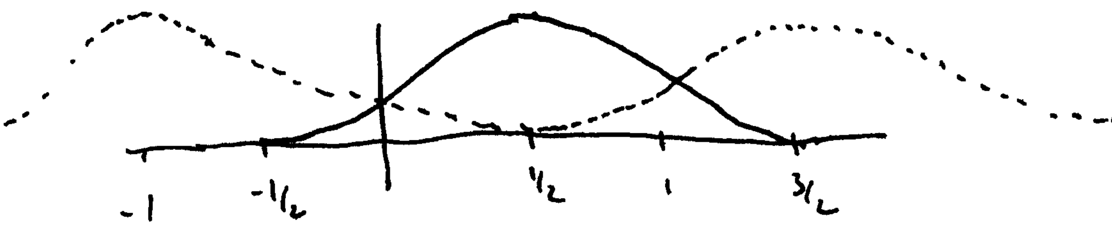
	$u_0\gamma_0$	A_0	B_0	B_0/A_0
4x oversampled	$\pi/2$	3.9	4.1	1.05
	$3\pi/4$	2.5	2.8	1.1
2x oversampled	π	1.6	2.4	1.5
	$4\pi/3$	0.58	2.1	3.6
almost critically sampled	1.9π	0.09	2.0	22

Tight frames

Tight frames with $g(t)$ smooth and compactly supported are easy.

Example:

Suppose $g(t)$ is supported on $[-1/2, 3/2]$



And say $\sum_{n=-\infty}^{\infty} |g(t-n)|^2 = A$

Take Fourier series on $L_2([n-1/2, n+3/2])$

Then $\{\phi_{n,k}(t) = g(t-n) e^{jkt\cdot\pi}\}_{n,k \in \mathbb{Z}}$

is a tight frame for $L_2(\mathbb{R})$

proof:

$$\sum_{n,k} |\langle f, \phi_{n,k} \rangle|^2 =$$

$$=$$

$$=$$

$$=$$