Some additional notes on frame operators

1. (a) A mapping $L : H \to G$ from a Hilbert space H into a Hilbert space G is called a **linear operator** if for all $\alpha, \beta \in \mathbb{C}$ and $f, g \in H$

$$L[\alpha f + \beta g] = \alpha L[f] + \beta L[g].$$

   (b) The **operator norm** of L is defined as

$$\|L\| := \sup\left\{\|L[f]\|_G : f \in H,\ \|f\|_H = 1\right\},$$

   where $\|\cdot\|_H$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle_H$ on H and similarly for $\|\cdot\|_G$.

   (c) If $H = \mathbb{C}^n$ and $G = \mathbb{C}^m$, both equipped with the standard Euclidean inner product, then L can be represented by an $m \times n$ matrix and $\|L\|$ is the maximum singular value of L. (In a slight abuse of notation, in the finite case we use L to denote both the linear operator and the matrix which captures the action of this linear operator. This should not cause any confusion, though.)

2. (a) The **adjoint** of a linear operator L is the unique linear operator $L^* : G \to H$ such that

$$\langle L[f], y \rangle_G = \langle f, L^*[y] \rangle_H$$

   for all $f \in H$ and $y \in G$. Proof that such a linear operator exists and is unique can be found in Chapter 7 of Young.

   (b) If $H = \mathbb{C}^n$ and $G = \mathbb{C}^m$, both equipped with the standard Euclidean inner product, then L^* is represented by the conjugate transpose of L.

   (c) For general Hilbert spaces, it is always true that $(L^*)^* = L$ and that $\|L^*\| = \|L\|$. We call $L : H \to H$ **self-adjoint** or **Hermitian** if $L = L^*$ (for $H = \mathbb{C}^n$, this means the matrix is conjugate-symmetric).

3. (a) We can define the range and the null space of a linear operator in much the same way as we do for matrices in finite dimensions:

$$\begin{aligned} \mathrm{Null}(L) &= \{f \in H :\ L[f] = 0\} \\ \mathrm{Range}(L) &= \{y \in G :\ y = L[f] \text{ for some } f \in H\}. \end{aligned}$$

   (b) As in finite dimensions:

$$\begin{aligned} &\text{if } y \in \mathrm{Range}(L) \text{ and } v \in \mathrm{Null}(L^*),\ \text{then } \langle y, v \rangle_G = 0; \\ &\text{if } f \in \mathrm{Range}(L^*) \text{ and } g \in \mathrm{Null}(L),\ \text{then } \langle f, g \rangle_H = 0. \end{aligned}$$

4. (a) Given a sequence of signals $\{\phi_k\}_{k\in\Gamma}$ in a Hilbert space H, we can define the linear operator $\Phi : H \to \ell_2(\Gamma)$ by

$$\Phi[f] = \{\langle f, \phi_k \rangle_H\}_{k\in\Gamma}.$$

   We call Φ the **frame operator** associated with the sequence $\{\phi_k\}_{k\in\Gamma}$, even though $\{\phi_k\}_{k\in\Gamma}$ may or may not be a frame (see below).

(b) As in the notes, we are using Γ to represent an arbitrary countable index set. If the sequence $\{\phi_k\}$ contains a finite number of elements n, then we might take $\Gamma = \{1, \ldots, n\}$ or $\Gamma = \{0, \ldots, n-1\}$ or maybe even some other set of n distinct integers depending on which indexing scheme is most natural. If $\{\phi_k\}$ contains an infinite number of elements, then we might take $\Gamma = \mathbb{Z}$ or $\Gamma = \{0, 1, 2, \ldots\}$ or maybe some other countable set, depending on which indexing scheme is more natural. For these discrete spaces, we will simply write ℓ_2 in place of $\ell_2(\Gamma)$ or $\mathbb{C}^n$ or $\mathbb{R}^n$ when the meaning is clear.

(c) When $H = \mathbb{C}^n$ and $|\Gamma| = m$ is finite, then Φ corresponds to an $m \times n$ matrix whose rows are the ϕ_k^*.

5. (a) The adjoint Φ^* of the frame operator Φ maps sequences of numbers to signals in H by taking the corresponding superposition of the ϕ_k; $\Phi^* : \ell_2(\Gamma) \to H$ by

$$\Phi^*[\alpha] = \sum_{k \in \Gamma} \alpha_k \phi_k.$$

(b) We will sometimes refer to Φ as the **analysis operator** and Φ^* as the **synthesis operator** corresponding to $\{\phi_k\}_{k \in \Gamma}$.

6. (a) If there exist real numbers $0 < A \le B < \infty$ such that

$$A\|f\|_H^2 \;\le\; \|\Phi[f]\|_{\ell_2}^2 = \sum_{k \in \Gamma} |\langle f, \phi_k \rangle_H|^2 \;\le\; B\|f\|_H^2,$$

then the $\{\phi_k\}_{k \in \Gamma}$ form a **frame** for H and A and B are called the **frame bounds**.

(b) If the ϕ_k are linearly independent, then $\{\phi_k\}_{k \in \Gamma}$ is a **Riesz basis** for H.

(c) If $\|\phi_k\|_H = 1$ for all k, then $A \le 1 \le B$.

(d) If $\|\phi_k\|_H = 1$ for all k and $A = B = 1$, then $\{\phi_k\}_{k \in \Gamma}$ is an **orthobasis** for H.

(e) If $A = B$, then $\{\phi_k\}_{k \in \Gamma}$ is a **tight frame** for H.

7. (a) At the very least, for $\{\phi_k\}_{k \in \Gamma}$ to qualify as a basis or a frame, it must be **complete**. That is, it must be true that

$$\text{if } \langle f, \phi_k \rangle_H = 0 \text{ for all } k \in \Gamma, \text{ then } f = 0.$$

In other words, there is no $f \in H$ that is orthogonal to all of the ϕ_k. An equivalent statement is that $\mathrm{span}(\{\phi_k\}_{k \in \Gamma})$ is **dense** in H; that is, for every $f \in H$ and every $\epsilon > 0$ there exists a $f' \in \mathrm{span}(\{\phi_k\}_{k \in \Gamma})$ such that $\|f - f'\|_H \le \epsilon$.

(b) If H has finite dimension n, then any sequence $\{\phi_k\}_{k \in \Gamma}$ is complete if at least n of the ϕ_k are linearly independent (so clearly we need $|\Gamma| \ge n$). We also do not have this technicality of "dense" in finite dimensions: it will simply be true that $H = \mathrm{span}(\{\phi_k\}_{k \in \Gamma})$.

(c) It is a little annoying that we are using the word "complete" here to mean something very different than on page I.8 of the notes. A norm/inner-product space being "complete" means something completely different than a sequence of signals within an inner product space being "complete". But so it goes.

(d) If $\{\phi_k\}_{k\in\Gamma}$ is not complete, then the lower frame bound $A = 0$. In finite dimensions, the converse is true: if $A = 0$, then $\{\phi_k\}_{k\in\Gamma}$ is not complete. In infinite dimensions, we can have complete $\{\phi_k\}_{k\in\Gamma}$ for which $A = 0$; see the examples on page I.33 of the notes.

(e) The sequence $\{\phi_k\}_{k\in\Gamma}$ is complete if and only if $\text{Null}(\Phi) = \{0\}$. This is equivalent to the closure of $\text{Range}(\Phi^*)$ being the entirety of H (i.e. $\text{Range}(\Phi^*)$ is dense in H).

8. Notice that

$$\|\Phi[f]\|_{\ell_2}^2 = \langle \Phi[f], \Phi[f]\rangle_{\ell_2} = \langle f, \Phi^*\Phi[f]\rangle_H,$$

where $\Phi^*\Phi : H \to H$ is a self-adjoint linear operator. We can use this fact to re-write the frame bounds as solutions to the following optimization programs:

$$A = \inf\{\ \langle f, \Phi^*\Phi[f]\rangle_H :\ \|f\|_H = 1\}$$
$$B = \sup\{\ \langle f, \Phi^*\Phi[f]\rangle_H :\ \|f\|_H = 1\}.$$

9. (a) Given the sequence $\{\phi_k\}_{k\in\Gamma}$ we can form the **Gram matrix** $\Phi\Phi^*$ with

$$(\Phi\Phi^*)_{j,k} = \langle \phi_j, \phi_k\rangle_H, \quad j,k \in \Gamma.$$

(b) If $\{\phi_k\}_{k\in\Gamma}$ is a finite sequence of length n, then $\Phi\Phi^*$ is an $n\times n$ matrix which is Hermitian and positive-semidefinite (i.e. has real eigenvalues that are non-negative).

(c) If $\{\phi_k\}_{k\in\Gamma}$ is an infinite sequence, then $\Phi\Phi^*$ is a "matrix" with an infinite number of rows and columns. More precisely, it is a linear operator from $\ell_2(\Gamma)$ into $\ell_2(\Gamma)$ whose action on $\alpha \in \ell_2(\Gamma)$ is given by

$$(\Phi\Phi^*[\alpha])_j = \sum_{k\in\Gamma} \langle \phi_j, \phi_k\rangle_H \alpha_k.$$

(d) We can relate the Gram matrix to the frame bounds A, B in the following way. If the sequence $\{\phi_k\}_{k\in\Gamma}$ is complete, then

$$A\|\alpha\|_{\ell_2}^2 \ \le\ \|\Phi^*\alpha\|_H^2 \ \le\ B\|\alpha\|_{\ell_2}^2 \quad \text{for all} \quad \alpha \in \text{Range}(\Phi).$$

Notice the condition $\alpha \in \text{Range}(\Phi)$; it cannot be ignored. There are plenty of perfectly good frames both in finite and infinite dimensions for which Φ^* has a null space; restricting the above to $\text{Range}(\Phi)$ only allows us to consider vectors orthogonal to this null space.

(e) Other ways to write $\|\Phi^*\alpha\|_H^2$ include

$$\|\Phi^*\alpha\|_H^2 \ =\ \left\|\sum_{k\in\Gamma} \alpha_k\phi_k\right\|_H^2 \ =\ \langle \Phi^*\alpha, \Phi^*\alpha\rangle_H \ =\ \langle \alpha, \Phi\Phi^*\alpha\rangle_{\ell_2},$$

and so if $\{\phi_k\}_{k\in\Gamma}$ is complete,

$$A = \inf\{\ \langle \alpha, \Phi\Phi^*\alpha\rangle_{\ell_2}\ : \alpha \in \text{Range}(\Phi),\ \|\alpha\|_{\ell_2} = 1\},$$
$$B = \sup\{\ \langle \alpha, \Phi\Phi^*\alpha\rangle_{\ell_2}\ : \alpha \in \text{Range}(\Phi),\ \|\alpha\|_{\ell_2} = 1\}.$$

Actually, we do not need the restriction $\alpha \in \text{Range}(\Phi)$ for B; we can take the supremum over all $\alpha \in \ell_2(\Gamma)$ that have unit norm (why?).

10. (a) If the $\{\phi_k\}_{k\in\Gamma}$ form a frame for H, we can recover any $f \in H$ from its expansion coefficients $\Phi[f] = \{\langle f, \phi_k\rangle_H\}_{k\in\Gamma}$. It is a fact that if the lower frame bound $A > 0$, then the self-adjoint linear operator $\Phi^*\Phi$ is invertible. We define the **dual frame** $\{\tilde{\phi}_k\}_{k\in\Gamma}$ as

$$\tilde{\phi}_k = (\Phi^*\Phi)^{-1}\phi_k.$$

(b) The sequence $\{\tilde{\phi}_k\}_{k\in\Gamma}$ is also a frame for H. The associated frame operator is denoted $\tilde{\Phi}$. For all $f \in H$,

$$\frac{1}{B}\|f\|_H^2 \;\le\; \|\tilde{\Phi}[f]\|_{\ell_2}^2 = \sum_{k\in\Gamma} |\langle f, \tilde{\phi}_k\rangle_H|^2 \;\le\; \frac{1}{A}\|f\|_H^2.$$

(c) We can re-write the expression $f = (\Phi^*\Phi)^{-1}\Phi^*\Phi[f]$ as the following **reproducing formula**:

$$f = \sum_{k\in\Gamma} \langle f, \phi_k\rangle_H \, \tilde{\phi}_k.$$

We can also switch the roles of the primal and dual frames:

$$f = \sum_{k\in\Gamma} \langle f, \tilde{\phi}_k\rangle_H \, \phi_k.$$

In the first reproducing formula above, $\{\phi_k\}_{k\in\Gamma}$ is the **analysis frame** and $\{\tilde{\phi}_k\}_{k\in\Gamma}$ is the **synthesis frame**. In the second reproducing formula, $\{\tilde{\phi}_k\}_{k\in\Gamma}$ is the analysis frame while $\{\phi_k\}_{k\in\Gamma}$ is the synthesis frame.

(d) If the $\{\phi_k\}_{k\in\Gamma}$ are a Riesz basis, then Φ itself is invertible, and $\tilde{\phi}_k = \Phi^{-1}\delta^k$, where δ^k is a sequence of numbers in $\ell_2(\Gamma)$ with $\delta^k_n = 1$ if $n = k$ and is zero elsewhere. In this case, it is easy to check that

$$\langle \phi_k, \tilde{\phi}_j\rangle_H = \begin{cases} 1 & k = j \\ 0 & \text{otherwise} \end{cases}.$$

Together, the sequences $\{\phi_k\}_{k\in\Gamma}$ and $\{\tilde{\phi}_k\}_{k\in\Gamma}$ are called **biorthogonal bases**.

(e) If the $\{\phi_k\}_{k\in\Gamma}$ are a tight frame ($A = B$), then $\Phi^*\Phi = AI$ where I is the identity operator on H. Then $\tilde{\phi}_k = \frac{1}{A}\phi_k$, and the reproducing formula becomes

$$f = \frac{1}{A}\sum_{k\in\Gamma} \langle f, \phi_k\rangle_H \, \phi_k.$$

(f) Notice that a tight frame analysis operator preserves the inner product (to within a scaling) as well as the norm:

$$\langle \Phi[f], \Phi[g]\rangle_{\ell_2} = \langle f, \Phi^*\Phi[g]\rangle_H = A\langle f, g\rangle_H \quad \text{for all} \quad f, g \in H.$$

11. A word of caution: while it is true for orthobases that $\Phi\Phi^* = AI$, this is **not true for tight frames in general**. For tight frames that are not orthobases, the synthesis operator $\frac{1}{A}\Phi^*$ has a non-trivial null space, and so norms and inner products in coefficient space are not necessarily preserved after synthesis. That is

$$\|\alpha\|_{\ell_2}^2 \ne A\|\Phi^*\alpha\|_H^2 \quad \text{for} \quad \alpha \notin \text{Range}(\Phi)$$
$$\langle \alpha, \beta\rangle_{\ell_2} \ne A\langle \Phi^*\alpha, \Phi^*\beta\rangle_H \quad \text{for general} \quad \alpha, \beta \notin \text{Range}(\Phi).$$