Frames

A **frame** is slightly more general than a Riesz basis — we remove the requirement of linear independence.

Definition: A frame for a Hilbert space H is a set $\{\phi_n\}_{n\in T}$ such that $\exists\ 0 < A \le B < \infty$ with

$$A \cdot \|f\|_H^2 \le \sum_{n\in T} |\langle f, \phi_n\rangle|^2 \le B \cdot \|f\|_H^2$$

for all $f \in H$.

Interesting fact: There exist frames which are not orthobases which have $A = B$ (a Parseval relation holds)

These are called **tight frames**.

Frame operator: $\Phi : H \to \ell_2(T)$ (infinite dimensions)

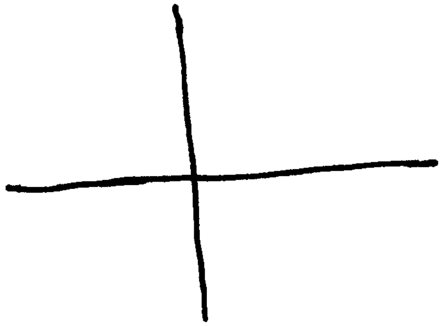
or $\Phi : \mathbb{C}^n \to \mathbb{C}^m$ (finite dimensions)

$\hookrightarrow$ just as before Φ has the ϕ_k^* as rows — it is an $m\times n$ matrix

Example: $H = \mathbb{R}^2$

$$\varphi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

Sketch:



Frame operator: $\Phi = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \sqrt{3}/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}$

$$\Phi^* \Phi = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$\Rightarrow \underline{\qquad\qquad} \le f^* \Phi^* \Phi f \le \underline{\qquad\qquad}$$

$$\Rightarrow A = B = \underline{\qquad\qquad}$$

Example 2: Unions of orthobases

$$\{\phi^1_{k_1}\}_{k_1 \in \Gamma_1} \cup \{\phi^2_{k_2}\}_{k_2 \in \Gamma_2} \cup \cdots \cup \{\phi^M_{k_M}\}_{k_M \in \Gamma_M}$$

Then

$$\|\Phi f\|^2 = \sum_{k_1 \in \Gamma_1} |\langle f, \phi^1_{k_1}\rangle|^2 + \sum_{k_2 \in \Gamma_2} |\langle f, \phi^2_{k_2}\rangle|^2 + \cdots + \sum_{k_M \in \Gamma_M} |\langle f, \phi^M_{k_M}\rangle|^2$$

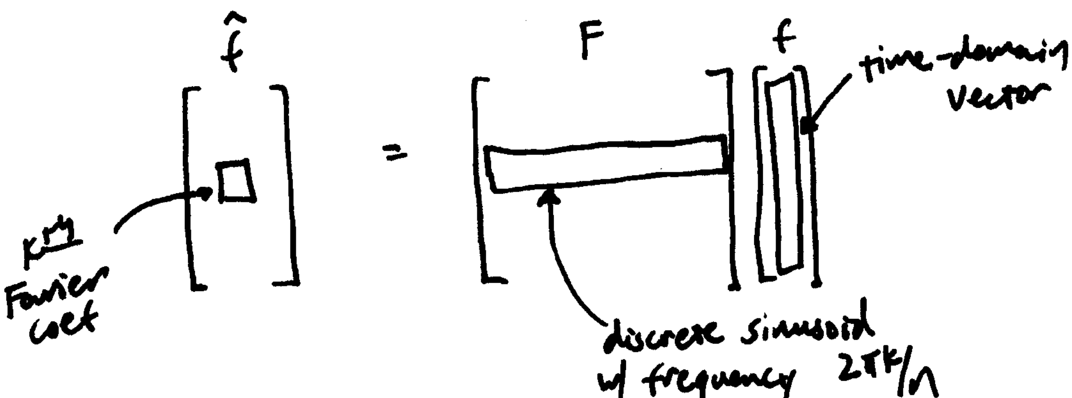
$$= \|f\|_H^2 + \|f\|_H^2 + \cdots + \|f\|_H^2$$

$$= M \cdot \|f\|_H^2$$

Example 3: Oversampled DFT

Recall that the $n \times n$ DFT matrix F is

$$(F)_{k+1, \ell+1} = \frac{1}{\sqrt{n}} e^{-j\frac{2\pi k \ell}{n}} \qquad \ell, k = 0, \ldots, n-1$$



We know very well by now that the rows of F form an orthobasis for Φ

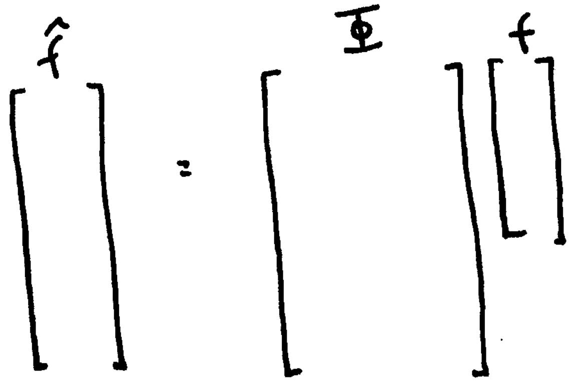
n sinusoids at equally spaced freqs. between 0 & 2π

We will look at an <u>oversampled</u> <u>DFT</u>,

$$(\Phi)_{k+1,\,\ell+1} = \frac{1}{\sqrt{n}}\, e^{-j\frac{2\pi k \ell}{m}} \qquad \begin{array}{l} \ell = 0, \ldots, n-1 \\ k = 0, \ldots, m-1 \end{array}$$

Rows are $m > n$ complex sinusoids with equally spaced freqs between 0 & 2π.

In MATLAB: fft(f, m) $\qquad$ length(f) $= n$



Now, $(\Phi^* \Phi)_{\ell'+1,\,\ell+1} = \langle \qquad , \qquad \rangle$

$$= \sum_{k=0}^{m-1}$$

$$= \Bigg\{$$

Thus

$$\sum_{k=1}^{m} |\langle f, \phi_k \rangle|^2 = f^* \Phi^* \Phi f$$

$$= \underline{\qquad\qquad}$$

General rule:

Oversampled systems are frames.
When the sampling is equispaced, the system is a tight frame

(Much more about this on Homework #1)

Note that for a tight frame
$$\Phi^*\Phi = A \cdot I$$

So we have a Parseval theorem:
$$\|\Phi f\|_2^2 = \langle \Phi f, \Phi f\rangle = \langle f, \Phi^*\Phi f\rangle$$
$$= A\langle f, f\rangle$$
$$= A \cdot \|f\|_2^2$$
energy is simply scaled by a constant

Also, inner products are preserved
$$\langle \Phi f, \Phi g\rangle = A\langle f, g\rangle$$

IMPORTANT NOTE:

It is NOT true that $\Phi\Phi^* = A \cdot I$

(It can't be, as it is a $m \times m$ matrix of rank n, $n < m$)

Reconstructing from frame coefficients

How do we go from $\{\langle f, \phi_k \rangle\}_{k\in\Gamma}$ back to f ?

Since the $\{\phi_k\}_{k\in\Gamma}$ are not linearly independent, given an arbitrary $\alpha \in \ell_2(\Gamma)$, there may not be an f with

$$\Phi f = \alpha$$

i.e. in finite dimensions, Φ is an $m\times n$ matrix

$$\begin{bmatrix} \\ \alpha \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & \Phi & \\ & & \end{bmatrix} \begin{bmatrix} f \end{bmatrix}$$

Can we solve this system for any α?
(Of course not.)

Instead, we will take our best shot, reconstructing with the pseudo inverse.

Pseudo Inverse

The pseudo inverse of a frame operator $\Phi : H \to \ell_2(\Gamma)$ is denoted $\Phi^{+} : \ell_2(\Gamma) \to H$, and satisfies:

① left inverse

$$\Phi^{+}\Phi f = f \qquad \forall f \in H$$

② minimum error

$$\Phi^{+}\alpha = \arg\min_{f} \|\alpha - \Phi f\|_{\ell_2(\Gamma)}$$

"Which signal has $\{\langle f, \phi_k \rangle\}_{k\in\Gamma}$ closest to α?"

It is not hard to show that

$$\Phi^{+} = \underbrace{(\Phi^{*}\Phi)^{-1}}\,\Phi^{*}$$

this exists if the frame bounds satisfy $0 < A \leq B < \infty$

Note: If $\{\phi_k\}_{k\in\Gamma}$ is <u>tight</u>, then

$$\Phi^{*}\Phi = A \cdot I$$

$$\Rightarrow \Phi^{+} = \frac{1}{A} \cdot \Phi^{*}$$

pseudo inverse = applying the adjoint (rescaled), just as in the orthobasis case

Reproducing formula

$$f = \Phi^{+}\Phi f = (\Phi^*\Phi)^{-1}\Phi^*\Phi f$$
$$= \sum_{k\in\Gamma} \langle f, \phi_k\rangle \tilde{\phi}_k$$

where
$$\tilde{\phi}_k = (\Phi^*\Phi)^{-1}\phi_k$$

If $\{\phi_k\}_{k\in\Gamma}$ is tight, $\tilde{\phi}_k =$ ________

and the RF looks almost the same as in the orthobasis case

$$f = \frac{1}{A}\sum_{k\in\Gamma}\langle f, \phi_k\rangle \phi_k$$

Conditioning

Arguing as in the non-orthogonal Riesz basis case

$$\frac{1}{B}\|f\|_H^2 \le \sum_{k\in\Gamma}|\langle f, \tilde{\phi}_k\rangle|^2 \le \frac{1}{A}\|f\|_H^2$$

If $\lambda_k = \langle f, \phi_k\rangle + \varepsilon_k$

and $\tilde{f} = \sum_{k\in\Gamma}\lambda_k\tilde{\phi}_k$

then $\|f - \tilde{f}\|_H^2 \le \frac{1}{A}\cdot\|\varepsilon\|_{\ell_2(\Gamma)}^2$

BUT (and this is huge) this is really a worst case bound — the best case would be $\varepsilon \in Null(\Phi^{+}) \Rightarrow \|f - \tilde{f}\|_H^2 = 0$

If the error is random, then it is an "average" of these two cases.

Recall: Gaussian random vectors

suppose $z \in \mathbb{C}^m$ is iid Gaussian

$$z(k) \sim N(0, \sigma^2)$$

then

$$E\|z\|_2^2 = m \cdot \sigma^2$$

Let A be an $n \times m$ matrix.

What is $E\|Az\|_2^2$?

$$E\|Az\|_2^2 = E\langle Az, Az\rangle = E\langle z, A^*Az\rangle$$
$$= E\langle z, U^*\Lambda U z\rangle = E\langle Uz, \Lambda U z\rangle$$

U: ⊥-normal
Λ: diagonal, ≥ 0

Set $z' = Uz$
z' is also iid Gaussian (since U ⊥-normal)

$$z'(k) \sim N(0, \sigma^2)$$

We have

$$E\|Az\|_2^2 = E\langle z', \Lambda z'\rangle$$

$$=$$

$$=$$

$$\boxed{=\qquad\qquad}$$

<u>Oversampling for noise reduction</u>

Signal $f \in \mathbb{C}^n$

Say we observe the DFT coefficients of f in the presence of noise, and then attempt to reconstruct f:

$$\Phi = \text{DFT} \quad (\text{orthonormal})$$

Observe: $y = \Phi f + \varepsilon \qquad \varepsilon(k) \sim N(0, \sigma^2)$

Reconstruct: $\tilde{f} = \Phi^* y = \Phi^*(\Phi f + \varepsilon)$

$$= f + \Phi^* \varepsilon$$

Mean-square error:

$$E\|f - \tilde{f}\|_2^2 = E\|\Phi^*\varepsilon\|_2^2 = n\sigma^2$$

Now, say we observe <u>oversampled</u> DFT coefficients in the presence of noise

Φ = oversampled DFT, $m \times n$, $m > n$

$$(\Phi)_{k+1,\ell+1} = \frac{1}{\sqrt{n}} e^{\frac{-j2\pi k\ell}{m}} \qquad \begin{array}{l} \ell = 0, \dots, n-1 \\ k = 0, \dots, m-1 \end{array}$$

We saw previously that Φ is a tight frame with $A = B = m/n$, i.e.

$$\Phi^* \Phi = \frac{m}{n} \cdot I$$

Observe: $y = \Phi f + \varepsilon \qquad (y \in \mathbb{C}^m,\ \varepsilon \in \mathbb{C}^n)$

Reconstruct w/ pseudo inverse:

$$\tilde{f} = (\Phi^* \Phi)^{-1} \Phi^* y = (\Phi^* \Phi)^{-1} \Phi^* (\Phi f + \varepsilon)$$

$$= f + \frac{n}{m} \Phi^* \varepsilon$$

$$\left(\text{since } (\Phi^* \Phi)^{-1} = \frac{n}{m} \cdot I\right)$$

Mean-squared error:

$$E\|f - \tilde{f}\|_2^2 = E\left\| \frac{n}{m} \Phi^* \varepsilon \right\|_2^2$$

$$= \left(\frac{n}{m}\right)^2 \cdot \operatorname{Trace}(\Phi \Phi^*) \cdot \sigma^2$$

$$= \left(\frac{n}{m}\right)^2 \cdot m \cdot \sigma^2$$

$$\boxed{= \left(\frac{n}{m}\right) \cdot n \cdot \sigma^2}$$

Since $n < m$, the reconstruction error is <u>smaller</u>.

$$\frac{1}{\text{oversampling}} = \text{noise reduction factor !}$$

A portion of the noise lies in the null space of Φ^*.

<u>Key application</u>: Oversampled A/D conversion

We can quantize samples more coarsely if we sample them faster.