Non-orthogonal Bases & Frames

Reference: Mallat, Section 5.1

We have seen that if $\{\theta_n\}$ is an orthobasis (complete orthonormal sequence) for a Hilbert space H, we can "recover" (using the reproducing formula) any $f \in H$ from the sequence $\{\langle f, \theta_n \rangle\}$ in the most stable way imaginable (thanks to Parseval)

The questions arise:
Given a general sequence of vectors $\{\theta_n\}$

① When can we recover f from

$$\{\langle f, \theta_n \rangle\}_n = \begin{bmatrix} \langle f, \theta_1 \rangle \\ \langle f, \theta_2 \rangle \\ \vdots \\ \langle f, \theta_k \rangle \\ \vdots \\ \vdots \end{bmatrix} \quad ?$$

② How do you do it?

③ How stable is the recovery?

We will break these questions into two cases:

ⓐ <u>Non-redundant (Riesz bases)</u>

The sequence $\{\phi_n\}$ is <u>linearly independent</u>.
Remove <u>any</u> of the ϕ_n from the sequence and it is <u>incomplete</u> (will not span H)
$\Rightarrow$ there is a <u>unique</u> way to represent each $f \in H$

ⓑ <u>Overcomplete (Frame)</u>

The sequence $\{\phi_n\}$ is complete and linearly <u>dependent</u>

$\Rightarrow$ there are an <u>infinite</u> number of ways to represent each $f \in H$.

The answer to ① is the same in both cases:
We can recover any $f \in H$ from $\{\langle f, \theta_n \rangle\}_{n \in \Gamma}$
if there exists
$$0 < A < B < \infty$$
such that
$$A \cdot \|f\|_H^2 \leq \sum_{n \in \Gamma} |\langle f, \theta_n \rangle|^2 \leq B \cdot \|f\|_H^2 \quad \forall f \in H$$
This is something akin to the Parseval condition:
we require that $\sum_{n \in \Gamma} |\langle f, \theta_n \rangle|^2$ is close to $\|f\|_H^2$

The ratio B/A is the <u>condition number</u> of the representation.

<u>Footnote:</u> We will start using Γ to represent a general discrete index set.
In finite dimensional space,
$$\Gamma = \{1, 2, \ldots, n\}$$
In infinite dimensional space, Γ is something which can be put in 1-to-1 correspondence with the integers, e.g.
$$\Gamma = \mathbb{Z}, \quad \Gamma = \mathbb{N}, \quad \Gamma = \mathbb{Z} \otimes \mathbb{Z}, \text{ etc.}$$
natural numbers

pairs of integers

Non-redundant systems (finite dimensions)

Let's start with $H = \mathbb{C}^n$ and a sequence of $\phi_1, \phi_2, \ldots, \phi_m \in \mathbb{C}^n$

$$f = \begin{bmatrix} f(1) \\ f(2) \\ \vdots \\ f(n) \end{bmatrix} \in \mathbb{C}^n \qquad \alpha = \{\langle f, \phi_k \rangle\}_{k=1}^m = \begin{bmatrix} \langle f, \phi_1 \rangle \\ \langle f, \phi_2 \rangle \\ \vdots \\ \langle f, \phi_m \rangle \end{bmatrix} \in \mathbb{C}^m$$

Taking $$\Phi = \begin{bmatrix} \phi_1^* \\ \hline \phi_2^* \\ \hline \vdots \\ \hline \phi_m^* \end{bmatrix}$$ (the ϕ_n^* as rows)

We see that the matrix $\Phi : \mathbb{C}^n \to \mathbb{C}^m$

$$\Phi f = \alpha = \{\langle f, \phi_k \rangle\}_{k=1}^m$$

is the basis operator.

For the rows to be linearly independent, we need $m \leq n$

To be able to recover general f, we need $m \geq n$

i.e. we need $m = n$ and Φ^{-1} to exist

Reproducing formula

$$f = \Phi^{-1}\Phi f \qquad \text{(duh)}$$

$$= \sum_{k=1}^{n} \langle f, \phi_k \rangle \tilde{\phi}_k$$

where $\tilde{\phi}_k$ are the columns of Φ^{-1}

Note:

$$\langle \tilde{\phi}_k, \phi_\ell \rangle = \begin{cases} 1 & k=\ell \\ 0 & \text{otherwise} \end{cases} \quad \left(\begin{array}{c}\text{because } \Phi\Phi^{-1} = I \\ \text{as well}\end{array}\right)$$

The pair $\{\tilde{\phi}_k\}_{k=1}^{n}$, $\{\phi_k\}_{k=1}^{n}$ is called a biorthogonal basis for $\mathbb{C}^n$.

Q1: In $\mathbb{C}^n$, given a biorthogonal basis $\{\phi_k\}$, $\{\tilde{\phi}_k\}$ what are the "best" A and B s.t.

$$A \cdot \|f\|^2 \le \sum_{k=1}^{n} |\langle f, \phi_k \rangle|^2 \le B \cdot \|f\|^2 \quad ?$$

Q2: What are the best $\tilde{A}, \tilde{B}$ s.t.

$$\tilde{A} \cdot \|f\|^2 \le \sum_{k=1}^{n} |\langle f, \tilde{\phi}_k \rangle|^2 \le \tilde{B} \cdot \|f\|^2 \quad ?$$

Recall: The matrix $\Phi^*\Phi$ is symmetric (Hermitian) and positive-definite, so its eigenvectors are _________ and its eigenvalues are ________________.

In otherwords, the eigenvalue decomposition looks like

$$\Phi^*\Phi = U\Sigma U^*$$

where U is orthonormal ($U^*U = UU^* = I$) and Σ is diagonal with

$$\Sigma_{k,k} = \sigma_k = k^{\text{th}} \text{ largest eigenvalue of } \Phi^*\Phi$$

Note:

$$\begin{aligned}\sum_{k=1}^{n} |\langle f, \phi_k \rangle|^2 &= \langle \Phi f, \Phi f \rangle \\ &= \langle f, \Phi^*\Phi f \rangle \\ &= \langle f, U\Sigma U^* f \rangle \\ &= \langle U^* f, \Sigma U^* f \rangle\end{aligned}$$

Set $\alpha = U^* f$, note $\|\alpha\|^2 = \|f\|^2$ (U, U^* are $\perp$).

Then

$$\sum_{k=1}^{n} |\langle f, \phi_k \rangle|^2 = \langle \alpha, \Sigma \alpha \rangle$$
$$= \sum_{k=1}^{n} \sigma_k |\alpha_k|^2$$

since $\sigma_k > 0$

$$\sigma_{min} \cdot \|f\|^2 \leq \sum_{k=1}^{n} \sigma_k |\alpha_k|^2 \leq \sigma_{max} \cdot \sum_k |\alpha_k|^2 = \sigma_{max} \|f\|^2$$

$A = \sigma_{min}$ (smallest eigenvalue of $\Phi^* \Phi$)

$B = \sigma_{max}$ (largest eigenvalue of $\Phi^* \Phi$)

Similarly,

$$\sum_{k=1}^{n} |\langle f, \tilde{\phi}_k \rangle|^2 = \langle (\Phi^{-1})^* f, (\Phi^{-1})^* f \rangle$$
$$= \langle f, \Phi^{-1} (\Phi^{-1})^* f \rangle$$
$$= \langle f, (\Phi^* \Phi)^{-1} f \rangle$$

Since $(\Phi^* \Phi)^{-1} = U \Sigma^{-1} U^*$

$\Rightarrow \tilde{A} = 1/\sigma_{max}$

$\tilde{B} = 1/\sigma_{min}$

Stability

Suppose now that we perturb the coefficients $\{\langle f, \phi_k \rangle\}$.

Set $\lambda_k = \langle f, \phi_k \rangle + \varepsilon_k$

$$\bar{f} = \sum_{k=1}^{n} \lambda_k \tilde{\phi}_k = f + \sum_{k=1}^{n} \varepsilon_k \tilde{\phi}_k$$

Error $e = f - \bar{f} = \sum_k \varepsilon_k \tilde{\phi}_k = \Phi^{-1} \varepsilon$ $\quad \left(\varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix} \right)$

$$\underbrace{\|e\|^2}_{\text{error in reconstruction}} = \|\Phi^{-1}\varepsilon\|^2 \le \frac{1}{A} \cdot \underbrace{\|\varepsilon\|^2}_{\text{total "noise" added to coeffs}}$$

If A is close to 1, the system is very stable (error does not "blow up")

Note: If $\|\phi_k\| = 1 \ \forall k$, then $A \le 1 \le B$.

Why? Prove on homework.....

Infinite dimensions

In infinite dimensions, we have the same basic framework, but with infinite sequences

$$\Phi : H \to \ell_2(\Gamma)$$

$$\Phi[f] = \{\langle f, \phi_k \rangle\}_{k \in \Gamma}$$

Again, if $\exists \quad 0 < A \leq B < \infty$

$$A \cdot \|f\|_H^2 \leq \sum_{k \in \Gamma} |\langle f, \phi_k \rangle_H|^2 \leq B \cdot \|f\|_H^2 \quad \forall f \in H$$

then we can reconstruct f from $\{\langle f, \phi_k \rangle_H\}_{k \in \Gamma}$

i.e. $\exists$ another sequence $\{\tilde{\phi}_k\}_{k \in \Gamma}$ s.t.

$$f = \sum_{k \in \Gamma} \langle f, \phi_k \rangle \tilde{\phi}_k$$

($\{\tilde{\phi}_k\}_{k \in \Gamma}$ is called the <u>Riesz dual basis</u> to $\{\phi_k\}_{k \in \Gamma}$)

and

$$\frac{1}{B} \cdot \|f\|_H^2 \leq \sum_{k \in \Gamma} |\langle f, \tilde{\phi}_k \rangle_H|^2 \leq \frac{1}{A} \cdot \|f\|_H^2$$

$\Rightarrow$ the reconstruction is <u>stable</u>

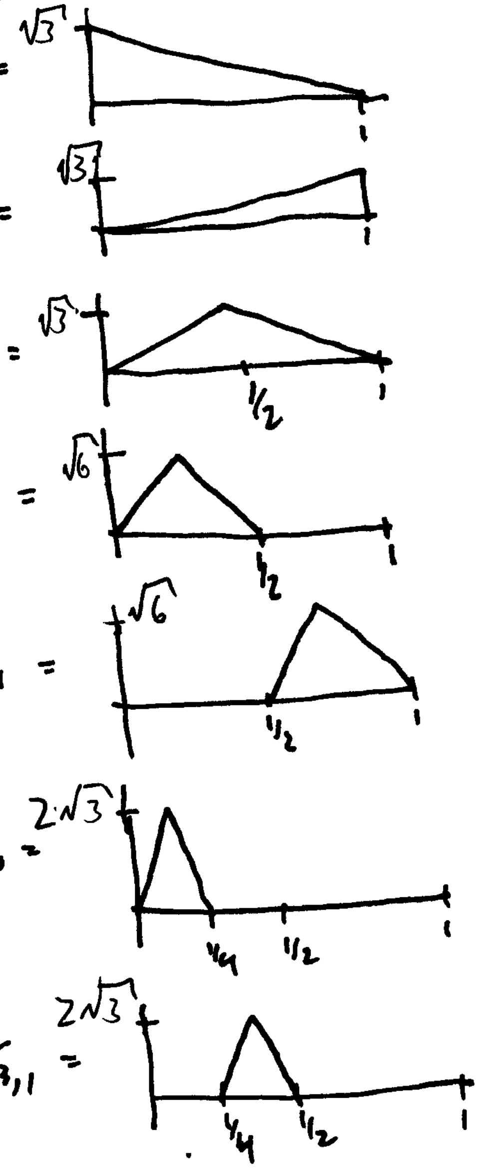
One thing changes in infinite dimensions:
We can have a "complete", linearly independent $\{\phi_n\}$ but have $A=0$.

<u>Example:</u> "Weighted Fourier Series" on $L_2([0,1])$

$$\{\phi_n\} = \left\{\frac{1}{n} e^{j2\pi n t}\right\}_{n\in\mathbb{Z}}$$

<u>Example w/ normalized ϕ_n:</u>

$\phi_{0,0} =$, $\phi_{0,1} =$, $\phi_{1,0} =$, $\phi_{2,0} =$, $\phi_{2,1} =$, $\phi_{3,0} =$, $\phi_{3,1} =$



⋮

If we cut this sequence of at N terms, the condition number

$$\frac{B}{A} \sim \sqrt{N}$$

(gets big)