ECE 3075, Fall 2008

Homework #8

Due Friday November 7, in class

Reading: Cooper and McGillem, Chapters 5 and 6

Using you class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better. Turn in your summary on a separate sheet of paper with your solutions to the problems below.

1. The file `hw8problem1.mat` contains a 2×543 matrix X. Each column of this matrix is an independent sample of a 2D Gaussian random vector $\underline{X}$.

   (a) Estimate the mean vector $\underline{\mu}$ of $\underline{X}$.

   (b) Estimate the covariance matrix K for $\underline{X}$.

   (c) Plot the contours of your estimated distribution on the same set of axes as a scatter plot of the data.

   (d) Use the MATLAB command `eig` to compute the eigenvalues and eigenvectors of the estimated covariance matrix K.

   (e) On the same set of axes as part (c), sketch the eigenvectors (offset by the mean vector).

2. Suppose that the two-dimensional random vector $\begin{bmatrix} X_1 & X_2 \end{bmatrix}^T$ has joint pdf

$$f_{X_1,X_2} = \begin{cases} 3/4 & 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq x_1^2 + 1 \\ 0 & \text{otherwise} \end{cases}.$$

   (a) What is the conditional pdf $f_{X_2}(x_2|X_1 = x_1)$?

   (b) Suppose we observe a particular value $X_1 = x_1$ for the first entry. What is the minimum mean-square estimate for the second value X_2?

   (c) Note that your answer to (b) is nonlinear. Now find the optimal *linear* (affine, really) estimate of X_2.

   (d) Plot your estimate for both parts (b) and (c) for all possible values of x_1. Put your plots on the same axes.

   (e) What is the mean-square error for each the estimators you found in (b) and (c)?

3. Answer "True" or "False". If true, explain why. If false, provide a specific counterexample.

   (a) If a random sequence $\{X_k\}$ is wide-sense stationary (WSS) then X_1 and X_2 are independent.

(b) If $\{X_k\}$ is WSS then $\mathrm{E}[X_1^2] = \mathrm{E}[X_2^2]$.

(c) If $\mathrm{E}[X_k]$ and $\mathrm{E}[X_k^2]$ do not depend on the time k, then $\{X_k\}$ is WSS.

4. Let $\{A_k\}$, $\{B_k\}$ be iid sequences of uniform random variables; $A_k \sim \text{Uniform}([-1,1])$, $B_k \sim \text{Uniform}([-1,1])$. Set

$$X_k = A_k(-1)^k + B_k(-1)^k$$

(a) Is the sequence $\{X_k\}$ stationary?

(b) Now suppose that both A_k and B_k are Uniform($[0,1]$). Does your answer to (a) change? Why?

5. Suppose $\{X_k\}$ is an iid sequence of Bernoulli random variables.

$$X_k = \begin{cases} 0 & \text{with prob } 1-p \\ 1 & \text{with prob } p \end{cases}.$$

(a) Form another sequence $Y_k = X_k + X_{k-1}$. Find the autocorrelation function $R_{YY}[m,n]$ and show that it is only a function of $m-n$.

(b) Do the same for $Z_k = X_k - X_{k-1}$.

6. You are playing Blackjack at a casino. When you win a hand, you get \$1; when you lose a hand, you give up \$1. Whether or not you win is independent between hands, and each hand has a $p = 0.5$ chance of winning. You start off with \$0 dollars (assume that the house will extend you credit if your losses exceed your winnings). The random sequence $\{X_k\}$ for $k = 1, 2, \ldots$ will represent your total earnings.

(a) What is mean function $\mu[k] = \mathrm{E}[X_k]$?

(b) What is the variance function $\sigma^2[k] = \mathrm{E}[X_k^2] - \mu[k]^2$?

(c) Is $\{X_k\}$ wide-sense stationary? Why or why not?

(d) Find the autocorrelation function $R_{XX}[m,n]$.

(e) Suppose that you have played 1000 hands. Estimate the probability that you have made at least \$50 and the probability you have lost at least \$50?

(f) Repeat (d) and (e) for the more realistic value of $p = 0.45$.

(For (e) and (f) above, note that the sum S_N of N independent Bernoulli random variables can be approximated by a Gaussian distribution whose mean and variance match that of S_N.)