Combinations

The following concepts come in useful when we are <u>counting</u> the number of possible outcomes for an experiment.

Consider a <u>population</u> (set) of size n
(think of balls labelled $1, \ldots, n$ in a big urn)

We draw an <u>ordered</u> <u>sample</u>

$$a_1, a_2, \ldots, a_r$$

consisting of r members of this population.

There are two ways we can do this:

① Sampling with replacement
After a ball is removed (and observed), you put it back in the urn.

→ For each of the r choices, there are n possibilities

→ There are

_____

different ordered samples which can be formed.

② Sampling without replacement

Once a ball is out, it stays out.

→ First sample, there are n possibilities
Second sample, there are $n-1$ possibilities,
etc.

→ There are

---

different ordered samples which
can be formed

Say we have a population of size n,
and we choose a sample of size r.
Fix a particular element of the population.
What is the probability of this element
being included in the sample for

① Sampling with replacement ?

② Sampling without replacement ?

Avoiding Duplicates

Suppose we draw a sample <u>with</u> <u>replacement</u> of size r from a population of size n. What is the probability that all r elements will be unique?

$$p = \frac{n}{n} \cdot \frac{(n-1)}{n} \cdot \frac{(n-2)}{n} \cdots \frac{(n-r+1)}{n}$$

$$= \frac{(n)_r}{n} \quad \text{where} \quad (n)_r = n \cdot (n-1) \cdots (n-r+1)$$

<u>Example 1 (Feller, Chap. 2)</u>

Draw a 5 digit number by selecting each digit uniformly at random from $\{0, 1, \ldots, 9\}$

The probability that all 5 are unique is

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{10^5} = 0.3024$$

Example 2 (Feller, chap 2)

n balls are placed randomly into n cells

The probability that every cell is occupied is

$$p = \frac{n!}{n^n}$$

For $n=6$, $p = 0.01543$

(roll a die six time, p = probability we see all six faces)

For $n=7$, $p = 0.00612$

(if there are 7 accidents/week on the downtown connector, almost all weeks will have a day with multiple accidents)

Subpopulations

Now suppose we sample w/o replacement, but we only care <u>which</u> balls we draw out, not what order they are in.

→ This is called the number of <u>subpopulations</u> of size r.

Suppose we have 6 balls. Then there are 15 subpopulations of size 2:

12	23	34	45	56
13	24	35	46	
14	25	36		
15	26			
16				

Compare to the $6 \cdot 5 = 30$ ordered samples that can be formed w/o replacement

Compare to the $6^2 = 36$ ordered samples that can be formed w/ replacement.

Clearly:

(# subpops. of size r) $\cdot$ (# possible orderings) = (# ordered samples w/o replacement)

$$\Rightarrow (\text{\# subpops. of size } r) = \frac{n!}{(n-r)!\,r!} = \binom{n}{r}$$

"n choose r"

Also called a <u>binomial coefficient</u>.

Note that

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} = \binom{n}{n-r}$$

It is also true that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Why?

Example

There are

$$\binom{52}{5} = 2{,}598{,}960$$

possible poker hands.

What is the probability of getting a flush (all same suit) on the deal?

There are $\binom{13}{5}$ spade flushes, $\binom{13}{5}$ heart flushes, etc.

so

$$\#\text{flushes} = 4 \cdot \binom{13}{5}$$

$$\Rightarrow \Pr(\text{flush}) = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{4 \cdot 13! \cdot 47! \cdot 5!}{5!\, 8!\, 52!}$$

$$= \frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = .0020$$

What is the probability that a hand contains 5 different face values?

Problem

You are playing draw poker, and have made it your mission to get a spade royal flush: As, Ks, Qs, Js, 10s

You get dealt 5 cards, you discard any not in the set above and replace them from the deck.

What is the probability that you will have the spade royal flush at the end?

Break this into 6 cases:

A: got none of the cards on the deal

B: got one of the cards on the deal

C: got two ——— " ———

D: got three ——— " ———

E: got four ——— " ———

F: got all five ——— " ———

(a) Calculate

$P(A) =$ $\quad$ $P(C) =$ $\quad$ $P(E) =$

$P(B) =$ $\quad$ $P(D) =$ $\quad$ $P(F) =$

(b) Use law of total probability

$$P(\text{flush}) = P(\text{flush}|A)\cdot P(A) + P(\text{flush}|B)\cdot P(B) + \cdots + P(\text{flush}|F)\cdot P(F)$$

Ordered Partitions

Let $r_1, r_2, \ldots, r_k$ be integers ≥ 0 such that

$$r_1 + r_2 + \cdots + r_k = n$$

How many ways can we arrange a population of size n into k ordered subgroups so that the first group contains r_1 elements, the second r_2, ..., the kth r_k?

Answer: $$\frac{n!}{r_1!\, r_2! \cdots r_k!}$$

(Note that the ordering of the groups is important as we are distinguishing between $(r_1 = 2, r_2 = 3)$ and $(r_1 = 3, r_2 = 2)$. But the order within the subgroups is ignored.)

Why is this true?

Select r_1 elements from all n

r_2 elements from remaining $n-r_1$

r_3 elements from remaining $n-r_1-r_2$

etc.

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdot\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-\cdots-r_{k-2}}{r_{k-1}}$$

$$= \frac{n!}{r_1!\,(n-r_1)!}\cdot\frac{(n-r_1)!}{r_2!\,(n-r_1-r_2)!}\cdot\frac{(n-r_1-r_2)!}{r_3!\,(n-r_1-r_2-r_3)!}\cdots$$

$$= \frac{n!}{r_1!\cdot r_2!\cdots r_k!}$$

<u>Example (Feller)</u>

In Bridge, 52 cards are divided into 4 hands, each of which goes to a particular player. There are

$$\frac{52!}{13!\cdot 13!\cdot 13!\cdot 13!} = 5.36\times 10^{28}$$

different situations.

What is the probability that each player has one ace?

The 4 aces can be arranged 4! different ways.

The remaining 48 cards can be arranged

$$\frac{48!}{12!\,12!\,12!\,12!}$$

different ways.

Thus

$$p = \frac{4!\;48!\;(13!)^4}{(12!)^4 \cdot 52!} \approx 0.105\cdots$$